Collusive Communication Schemes in a First-Price Auction

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Abstract

We study optimal bidder collusion at first-price auctions when the collusive mechanism only relies on signals about bidders' valuations. We build on Fang and Morris (2006) when two bidders have low or high private valuation of a single object and additionally receive a private noisy signal from an incentiveless center about their opponents' valuations.

We investigate the general case when the signals are chosen conditionally independently and identically out of $n \geq 2$ possible values. We demonstrate a symmetric equilibrium of the first price auction with *public* or *private* signals. We characterize the signal structure which provides the least revenue for the seller for arbitrary n. As a corollary, we show that bidders are *strictly* better off as signals can take on more and more possible values. Nevertheless, we provide an example which shows that the seller's revenue drops below the above optima even with only 2, but correlated signals.

Finally, we show that in case the center does not know the bidders' valuations, bidders have no incentives to report their types truthfully to the center. As a corollary: no collusive cheap talk equilibrium exists. In this sense the first price auction turns out to be collusion-proof.

Keywords: First price auction; Public and private signals; Collusion; Bidder-optimal signal structure; Cheap talk; Communication equilibrium; Collusion-proof mechanism.

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1 Introduction

In a standard IPV, first price auction it is assumed that each bidder observes her own valuation and has no information about her opponent's valuation except for the distribution from which it is drawn. However, for collusive purposes, bidders may agree and commit to a mechanism which provides them noisy information about each others' valuation.[1] Or there can exists an incentive compatible mechanism under which bidders may share information. With such multidimensional information bidders could improve their position against the seller and expect higher payoffs than with one-dimensional information only about their own valuation.[2] Fang and Morris (2006), Bergemann and Västimäki (2006) and Kim and Che (2004) are examples when the revenue equivalence of standard one-dimensional symmetric IPV auctions breaks down in the presence of such a signal mechanism. Our goal is to examine the extent to which bidders can improve their payoffs in a one-shot first price auction when they have access to such mechanisms.

Hence, we assume that two bidders have access to a center, an incentiveless third party which knows the realized valuation profile and can send random private or public signals to the bidders.[3] Thus, for a moment, we abstract away from the adverse selection problem, when bidders may not tell their true valuations to the center. We relax this assumption later and show how the center can or cannot elicit the necessary information from the bidders. Among others McAfee and McMillan (1992) and Marshall and Marx (2007) investigate optimal collusion with adverse selection. However, they allow the center to enforce side-payments and/or to enforce bids which may depend on the reports about the valuations.[4] We consider a one-shot auction, where there are no transfers and bids cannot be enforced by the center. That is, after learning their valuations and the signal sent by the center, bidders bid as they want. We concentrate on the question: what is the bidders' optimal signal structure that the center should use to increase

[1] Collusion in auctions has been investigated in many possible setups: repeated auction, costly bidding, resale, bribing etc.

[2] We assume, that the seller's behavior is passive. It is clear, that such behavior on the part of the seller is not the best response against a collusion. Therefore, we assume that the seller either does not know that the bidders have access to extra signals or is bound to the rules of the auction by law.

[3] Forges (2006) names such a center as an omniscient mediator.

[4] We elaborate on the connection between these papers and ours in the discussion.

bidders' payoffs?[5]

We show that in a first price auction with two bidders the collusive mechanism is better, the larger the variety of the noisy signals that bidders receive about the opponent's valuation.

We consider the simplest setup, where two bidders valuations are independently drawn from two possible valuation types $\{V_l, V_h\}$ and each bidder receives an extra signal, called information type, about the opponent's valuation. When the signal can only take values L or H, Fang and Morris (2006) show that, in the unique symmetric equilibrium, the seller expects less revenue in the first price than in the second price auction and both auctions allocate the object efficiently. We show that the revenue gap between the two auctions increases in the number of possible values the signals can take on. For example, if the signal can take on values from the set $\{L, M, H\}$ the equilibrium payoff of bidder is larger compared to the 2-valued signal case. Intuitively, it is clear that if the signal probabilities are chosen optimally more possible values can do no harm to the bidders, however it is ambiguous whether adding a new possible value strictly helps the bidders or not. We answer this question analytically when signals are drawn independently and identically for the bidders. We give a constructive method how to embed a new value into any given finite-valued signal structure in a way that bidder's payoff increases. However, an example with 2-valued but correlated signals is provided which outperforms the numerically calculated $n-$valued independent signal case for any large n. Finally, we characterize some important features of the optimal signal structure for arbitrary $n-$valued independent signal.

Finally, we address the question whether bidders can share information about their own valuation in an incentive compatible manner. We ask the question whether the bidders are willing to report truthfully their valuation to the center, who in turn gives hints (signals) to the bidders how to bid. The answer is negative. We show that the high type bidder always has incentive to lie and induce his opponent to bid less aggressively. We can interpret this result as an evidence for the first price auction being collusion proof. In particular, our result implies that bidders cannot share information in an equilibrium by using plain, unmediated cheap talk messages.

[5]Several papers (Bergemann and Pesendorfer, 2007; Eső and Szentes, 2007) investigate the same question from the seller's point of view. That is, how should the seller optimally disclose information about valuations.

Our results also complement the industrial organization literature on trade associations. It has been recognized that trade associations, through information sharing, can serve as collusion facilitating devices. (See, for example, Vives (1990) and the references in his Section 2.) For example, Genesove and Mullin (1997) provide an interesting case study of the workings of the Sugar Institute, the trade association uniting the U.S. domestic sugar refiners from 1928 to 1936. They describe how the Sugar Institute collected, aggregated and disseminated the data about the industry among its members, focusing in particular on the incentive issues. One of their findings is that "the Sugar Institute revealed less information to its members than it knew".[6] This conforms with our result that revealing noisy (public) information about the opponent's valuation increases each bidder's expected payoff. On the other hand, another finding by Genesove and Mullin (1997) that the Institute members did not misreport their private information cannot be explained within our framework. Our results indicate that other factors like auditing of accounting records must have played an important role in inducing honest revelation of information.

The paper unfolds as follows. In section 2, we set up the model. In section 3, we prove the existence of a symmetric equilibrium with private signals. In section 4, we state the theorem, calculate the optimal signal structure for 2-valued private signals and show how to improve bidders payoff when a third value is introduced. In Section 5 we present similar results for the public signal case. In Section 6, we show that no credible communication is possible between the bidders even if they have access to a third party. In Section 7, we present the 2-valued correlated signal case, provide some conjectures which we were unable to prove and argue that the first price auction may not be collusion-proof if we relax the bidders incentive constraints in an intuitive manner. Some proofs are relegated to the Appendix.

2 The Model

Two bidders, 1 and 2, compete for an object. When we refer to a generic bidder we use she and we do not index the notation if it does not cause confusion. Bidders' valuations of the object are private and independently drawn from identical distributions. We assume that bidder's valuation of the

[6]Genesove and Mullin (1997), p. 20.

object takes on two possible values $\{V_l, V_h\}$, where we set $V_l = 0, V_h = 1$.[7] The ex ante probability that a bidder's valuation v takes value V_l is denoted by $p \in (0,1)$. Of course, the probability of V_h is $1-p$.

As in standard private value auction models, bidders observe their own private valuation v. Fang and Morris (2006) assumes that each bidder also *privately* observes a noisy signal s about her opponent's valuation. For tractability, they assume that the noisy signal can take on only two possible qualitative categories $s \in \{L, H\}$.[8] The novel feature of this paper is that signals can take on values from the set $n = \{1, 2, \ldots, n\}$, and we also consider the case when bidders *publicly* observe the signals. Bidder 1's signal $s_1 \in n$ about 2's valuation v_2 is generated as follows. For all $j \in n$

$$\Pr(s_1 = j \mid v_2 = V_l) = x_j, \;\; \Pr(s_1 = j \mid v_2 = V_h) = y_j.$$

Bidder 2 receives s_2 about v_1 identically and independently of s_1. To sum up, when signals are sent *privately* bidder 1's type is a 2-tuple (v_1, s_1), where v_1 is bidder 1's valuation and s_1 is the signal about 2's valuation. In the case when signals are *public*, bidder 1's type is a 3-tuple (v_1, s_1, s_2), where v_1 is bidder 1's valuation, s_1 is the signal about 2's valuation and s_2 is the signal about bidder 1's own valuation. Of course, $\sum_{j\in n} x_j = \sum_{j \in n} y_j = 1$, $0 \le x_j \le 1$, and $0 \le y_j \le 1$ for all $j \in n$. We call $(x,y)_n = (x_j, y_j)_{j\in n}$ a *private* or *public signal structure*. We assume that there is no $j \in n$ such that $x_j = y_j = 0$. That is, each signal $j \in n$ has ex ante positive probability to appear, otherwise we are back to the case with less signals. Without loss of generality we assume that

$$\frac{x_1}{y_1} > \frac{x_2}{y_2} > ... > \frac{x_n}{y_n}.$$

If this relationship is not satisfied, we can always rename the signals. That is, we can always name the signal with the highest ratio as 1, and so on.[9] We prove later that if $\frac{x_j}{y_j} = \frac{x_{j+1}}{y_{j+1}}$ for some j then signals j and $j+1$ can be considered as one single signal with probabilities $x_{j'} = x_j + x_{j+1}, y_{j'} = y_j + y_{j+1}$ and the corresponding symmetric equilibria are the same in terms of the payoffs. Therefore, we will maintain these assumptions for the rest of the paper.

[7] All results extend in the obvious way for any $0 \le V_l < V_h$.

[8] In their set up, bidders may have 3 possible valuations. In this case, dealing with 3-valued signals is indeed untractable.

[9] If $y_j = 0$ then we set $\frac{x_j}{y_j} = \infty$.

3 An Equilibrium with Private Signals

First, we are interested in a symmetric Bayesian Nash equilibrium of the first price auction with zero reserve price,[10] where bidders simultaneously submit bids b depending on the realization of v and s. The highest bidder gets the object and pays her bid to the seller. In the event of a tie, the bidder with higher valuation gets the object and the tie-breaking can be arbitrary if bidders' valuations are the same.[11]

Proposition 1 [12]*A symmetric equilibrium of the first price auction with* private signal structure $(x, y)_n$ *is as follows:*

1. *Bidder of type* $(v, s) = (0, j)$ *bids 0, for any* $j \in n$. *Let* $\bar{b}_0 \equiv 0$.

2. *Bidder of type* $(v, s) = (1, 1)$ *bids 0 if* $y_1 = 0$, *otherwise she randomizes over* $\left[\bar{b}_0, \bar{b}_1\right]$ *according to the cumulative distribution function*

$$F_1(b) = \frac{px_1}{(1-p)\,y_1^2}\frac{b}{1-b},$$

where

$$\bar{b}_1 = 1 - \frac{px_1}{px_1 + (1-p)\,y_1^2}.$$

3. *Bidder of type* $(v, s) = (1, j)$, *for* $j = 2, ..., n$, *randomizes over* $\left[\bar{b}_{j-1}, \bar{b}_j\right]$ *according to the cumulative distribution function*

$$F_j(b) = \frac{px_j + (1-p)\,y_j \sum_{k=1}^{j-1} y_k}{(1-p)\,y_j^2}\frac{b - \bar{b}_{j-1}}{1-b}, \qquad (1)$$

where

$$\bar{b}_j = 1 - \frac{px_j + (1-p)\,y_j \sum_{k=1}^{j-1} y_k}{px_j + (1-p)\,y_j \sum_{k=1}^{j} y_k}\left(1 - \bar{b}_{j-1}\right). \qquad (2)$$

Since in equilibrium types $(0, j)$ bid 0 we suppress j and refer to them as type $v = 0$. For types $(1, j)$ we refer as type $s = j$.

[10]The assumption of zero reserve price is made purely for simplicity, and all subsequent results extend in the natural way if a binding reserve price is introduced.

[11]For the justification of this tie-breaking rule see Kim and Che (2004), Fang and Morris (2006) and Maskin and Riley (2000).

[12]All the major proofs are relegated to the Appendix.

Remark 1 *If for some* $j \in n, \frac{x_j}{y_j} = \frac{x_{j+1}}{y_{j+1}}$ *then* j *and* $j+1$ *have the same expected payoff for any bid in* $[\bar{b}_{j-1}, \bar{b}_{j+1}]$*. Moreover, it is easy to see that if we replace signals* j *and* $j+1$ *with* j' *having probabilities* $x_{j'} = x_j + x_{j+1}, y_{j'} = y_j + y_{j+1}$ *then in the corresponding equilibrium with* $n-1$ *signals it is true that* $\bar{b}_{j'} = \bar{b}_{j+1}$ *and the strategies of types different from* j' *do not change. This shows that our original assumption with strict inequalities is indeed without loss of generality.*

4 A Theorem for Private Signal Structures

Type 0 obtains zero expected payoff and type $j \in n$ obtains expected payoff:

$$K(j,(x,y)) = \frac{px_j + (1-p)\,y_j \sum_{k=1}^{j-1} y_k}{px_j + (1-p)\,y_j}\left(1 - \bar{b}_{j-1}\right). \tag{3}$$

Thus, each bidder's ex ante expected payoff is:

$$P(x,y) = (1-p)\sum_{j\in n}(px_j + (1-p)y_j)K(j,(x,y)).$$

Definition 1 *Given* n*, a* signal structure $(x,y)_n$ is optimal *for the bidders if it maximizes* $P(x,y)$*.*

Remark 2 *For any* $n \geq 2$ *there exists an optimal signal structure.*

Example 1 *Let* $n = 2$ *and* $x_1 = y_2 = q$ *and* $x_2 = y_1 = 1-q$*. That is, when signal 1 is equally indicative of value* V_l *as signal 2 is of value* V_h*. Then the equilibrium described in Proposition 1 is the same as in Proposition 1 of Fang and Morris (2006). They show that among these special signal structures there is an optimal* q *which minimizes the seller's revenue. However, this signal structure cannot be optimal for any* $q \in [0,1]$ *for the bidders. Indeed, the expected payoff of bidder when there are 2 signals is*

$$\begin{aligned} P(x,y) &= (1-p)px_1\left(1 + \frac{px_2 + (1-p)\,y_2 y_1}{px_1 + (1-p)\,y_1^2}\right) \\ &= (1-p)px_1\frac{p + (1-p)\,y_1}{px_1 + (1-p)\,y_1^2}. \end{aligned}$$

The first order condition w.r.t. x_1 *is*

$$\frac{p\left(1-p\right)^{2} y_1^2 \left(p+\left(1-p\right) y_1\right)}{\left(px_1+\left(1-p\right) y_1^2\right)^2},$$

which is strictly positive if $y_1 > 0$.[13] *Therefore, it is optimal to set* $x_1 = 1$. *The first order condition w.r.t.* y_1 *is*

$$\left(1-p\right) px_1 \frac{-\left(\left(1-p\right) y_1+p\right)^2+p\left(p+\left(1-p\right) x_1\right)}{\left(px_1+\left(1-p\right) y_1^2\right)^2},$$

which when set equal to 0 *and imposing* $x_1 = 1$ *implies that*

$$y_1 = \frac{\sqrt{p}}{1+\sqrt{p}}.$$

The second derivative w.r.t. y_1 *is negative when evaluated at the optimal value of* y_1. *This indicates that, if signals can take on 2 different values,* $x_1 = y_2 = q$ *and* $x_2 = y_1 = 1-q$ *does not hold under the optimal signal structure.*

Bidder's payoff, given the optimal signal structure, is

$$\left(1-p\right) \frac{\sqrt{p}\left(1+\sqrt{p}\right)}{2}.$$

For example, when $p = 0.25$, *bidder's ex ante payoff is* 0.2813, *while the seller's revenue is* 0.3750. *For comparison, Fang and Morris (2006) show that* q *that minimizes the seller's revenue is* 0.7639, *which results in the bidder's payoff of* 0.2628, *and the seller's revenue of* 0.4119.

Theorem 1 *An extra signal improves the bidder's optimal payoff:*

$$P((x,y)_n^\star) < P((x,y)_{n+1}^\star).$$

See the proof in the Appendix. To prove the theorem we take an arbitrary $(x,y)_n$ and we show that if $x_n > 0$ then it cannot be optimal. Hence, we assume that $x_n = 0$ and show that we can always introduce an additional signal value that strictly improves the bidder's payoff.

[13] If $y_1 = 0$, then $P\left(x,y\right) = p\left(1-p\right)$, that is, the same payoff as in the case when there are no signals at all.

Example 2 *To illustrate the approach adopted in the proof of the theorem, consider the optimal 2-valued signal structure found in Example 1. We construct the 3-signal structure by reassigning the probabilities as follows:*

$$\begin{aligned} x_1' &= (1-k)\,x_1, & y_1' &= (1-k)\,y_1, \\ x_2' &= kx_1, & y_2' &= ky_1 + \epsilon, \\ x_3' &= 0, & y_3' &= 1 - y_1 - \epsilon, \end{aligned}$$

Choosing $k = 0.5$ and $\epsilon = 0.1$, we find that bidder's ex ante payoff is 0.2841, *while the seller's revenue is* 0.3693 *when* $p = 0.25$. *Hence, the bidder's payoff is strictly higher than the one obtained under the optimal 2-valued signal structure.*

Further, the optimal 3-valued signal structure is[14]

$$\begin{aligned} x_1 &= 0.7295, & y_1 &= 0.1941, \\ x_2 &= 0.2705, & y_2 &= 0.1941, \\ x_3 &= 0, & y_3 &= 0.6118, \end{aligned}$$

which leads to the bidder's payoff of 0.2884, *and the seller's revenue of* 0.3607.

5 Public Signal Structures

Similar results can be established when the signals about the valuations are public. First, we determine what are the equilibrium strategies. Next, we derive the expressions for the equilibrium payoffs and show that each bidder's ex ante payoff increases with an extra signal. It is important to emphasize that we keep the original interpretation of the signals. For example, s_1 still represents an imperfect signal about the valuation of *bidder 2*.

Proposition 2 *Given public signals $(s_1, s_2) = (k, l)$ such that $k \leq l$, an equilibrium of the first price auction is as follows:*[15]

1. *A bidder with $v = 0$ bids 0;*

2. *Bidder 1 with $v_1 = 1$ randomizes according to*

$$F_{lk}(b) = \frac{px_k}{px_k + (1-p)\,y_k} \frac{px_l + (1-p)\,y_l}{(1-p)\,y_l} \frac{1}{1-b} - \frac{px_l}{(1-p)\,y_l} \tag{4}$$

   on the interval $\left[0, \bar{b}_k\right]$;

[14] The optimum was calculated numerically.

[15] If $y_1 = 0$ and $k = 1$, both bidders bid 0.

3. Bidder 2 with $v_2 = 1$ randomizes according to

$$F_{kl}\left(b\right) = \frac{px_k}{\left(1-p\right)y_k}\frac{b}{1-b} \tag{5}$$

on the interval $\left[0, \bar{b}_k\right]$, where

$$\bar{b}_k = \frac{\left(1-p\right)y_k}{px_k + \left(1-p\right)y_k}.$$

The equilibrium payoff of a bidder with valuation $v = 1$ is

$$\pi_k \equiv \frac{px_k}{px_k + \left(1-p\right)y_k}. \tag{6}$$

Clearly, the equilibrium when $k > l$ is obtained by interchanging the strategies of bidders 1 and 2 in Proposition 2. To summarize, the expected payoff of high valuation bidder, given any two public signals k and l, is given by $\pi_{\min\{k,l\}} = \max\left\{\pi_k, \pi_l\right\}$. Therefore, the ex ante payoff of bidder i can be written as

$$R\left(x,y\right) = \left(1-p\right)u'\Pi y$$

where

$$\Pi = \begin{pmatrix} \pi_1 & \pi_1 & \pi_1 & \cdots & \pi_1 & \pi_1 \\ \pi_1 & \pi_2 & \pi_2 & \cdots & \pi_2 & \pi_2 \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_3 & \pi_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_{n-1} & \pi_{n-1} \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_{n-1} & \pi_n \end{pmatrix}_{n\times n},$$

$u = \left(u_1, ..., u_n\right)'$, and $u_k = px_k + \left(1-p\right)y_k$ for $k = 1, ..., n$. In the matrix Π, a row k corresponds to signal $s_i = k$ about the valuation of bidder $j \neq i$, while a column l corresponds to signal $s_j = l$ about the valuation of bidder i, given that bidder i knows that she has high valuation. From the matrix Π and the definition of π_k it is again clear that if $x_k/y_k = x_{k+1}/y_{k+1}$ for some k, then signals k and $k+1$ can be merged in a single signal with the corresponding probabilities $x_k + x_{k+1}$ and $y_k + y_{k+1}$. Therefore, we again restrict attention to the case when $x_k/y_k > x_{k+1}/y_{k+1}$ for all $k = 1, ..., n-1$.

Example 3 *When $n = 2$, the ex ante payoff of bidder is*

$$R(x,y) = p(1-p)\left\{1 - y_1 + \frac{x_1 y_1}{px_1 + (1-p)y_1}\right\}. \tag{7}$$

Differentiating the curly brackets in (7) w.r.t. x_1 gives

$$\frac{(1-p)y_1^2}{(px_1 + (1-p)y_1)^2} > 0$$

therefore it is optimal to set $x_1 = 1$.[16] *Differentiating the curly brackets in (7) w.r.t. y_1, with $x_1 = 1$, gives*

$$-1 + \frac{p}{(p + (1-p)y_1)^2} = 0$$

or

$$y_1 = \frac{\sqrt{p}}{1+\sqrt{p}}.$$

Note that the solution for the optimal x_1 and y_1 is exactly the same as in the Example 1 with 2 private signals. The ex ante payoff of bidder evaluated at the optimal values is

$$2p(1-\sqrt{p}),$$

which is always lower than the ex-ante payoff with two private signals for all p.

When $n = 3$ and $p = 0.25$, the optimal 3-public signal structure is[17]

$$\begin{array}{ll} x_1 = 0.7130, & y_1 = 0.1975, \\ x_2 = 0.2870, & y_2 = 0.1975, \\ x_3 = 0, & y_3 = 0.6051, \end{array}$$

which is different from the optimal 3-private signal structure in Example 2. Note, however, the optimal structure possesses the same properties: $x_3 = 0$ and $y_1 = y_2 < y_3$. The bidder's ex-ante payoff is 0.2533, which is again lower than in the case of private signals. On the other hand, the extra signal improves on the bidder's payoff with 2 public signals, in which case the payoff is 0.25.

Theorem 2 *An extra signal improves the bidder's optimal payoff:*

$$R((x,y)_n^\star) < R((x,y)_{n+1}^\star).$$

[16] Again, if $y_1 = 0$, then the payoff is the same as in the case when there are no signals at all.

[17] Again, the optimum was calculated numerically.

6 The impossibility of credible communication

Up to now we have assumed that the center knows the valuations of the bidders and sends them signals (depending on the valuation profile) to which bidders freely peg their strategic bids. The adverse selection problem is the following. Suppose the center does not know the valuation profile and has to elicit it from the bidders to be able to generate the signals conditional on this information. In the following subsections we prove that the center is unable to elicit the bidders private information (their valuation). It turns out that independently of the signal structure, no matter whether it is public or private, the high type bidder would lie about her true valuation so as to induce a distribution of signals under which her opponent bids (on average) less aggressively. The main message of this section is that the first price auction turns out to be a collusion-proof mechanism.

Now we setup the extended game, where first, bidders are informed about their valuation and then may send a cheap talk message to the center, who in turn randomizes according to a given signalling structure and sends these signals privately or publicly to the bidders. Bidders submit bids which may depend on the signals and receive their payoffs according to the original first price auction. We show that there is no signalling structure and Bayes-Nash equilibrium of this extended game in which bidders expect a better payoff than in the non-cooperative equilibrium of the standard IPV first price auction. In particular, this result implies that bidders cannot collude on this first price auction using only *direct*, unmediated (without the center) cheap talk messages.

6.1 Communication Equilibrium

By the revelation principle Myerson (1982) it is sufficient[18] to consider canonical communication devices (centers), introduced also in Forges (1986), which receive type information from the bidders and depending on the reported type profile randomize according to a given rule q over the bid profiles and recommend these bids privately to the bidders. q is a canonical communication equilibrium if bidders sincerely report their types and obediently follow the recommended bids in Bayes-Nash equilibrium of the extended game. First

[18] See Cotter (1991) for the generalization when action and state spaces can be large.

we define a general communication equilibrium of the FPA. The *extended FPA* is as follows. Let $q : \{0,1\}^2 \to \Delta[0,1]^2$ be a canonical communication device:

1. bidder i sends private report $\bar{v}_i$ to the center from message space $\{0,1\}$;
2. the center selects a bid profile (b_i, b_{-i}) according to $q(\bar{v}_i, \bar{v}_{-i})(b_i, b_{-i})$;
3. the center privately informs bidder i about b_i;
4. bidders submit arbitrary bids at the FPA.

Denote with $FPA_i(v_i, v_{-i}, b_i, b_{-i})$ the expected payoff of bidder i with type v_i when submitting bid b_i while bidder $-i$ has valuation v_{-i} and has submitted b_{-i}. The expected utility of bidder i with valuation $v_i \in \{0,1\}$ when reporting $\bar{v}_i$ and after recommendation b_i bidding according to a measurable $r : [0,1] \to [0,1]$ given that bidder $-i$ is sincere and obedient is:

$$U_i(v_i, \bar{v}_i, r) = p \int_{b_i, b_{-i}} FPA_i(v_i, 0, r(b_i), b_{-i}) dq(\bar{v}_i, 0)(b_i, b_{-i}) +$$

$$+(1-p) \int_{b_i, b_{-i}} FPA_i(v_i, 1, r(b_i), b_{-i}) dq(\bar{v}_i, 1)(b_i, b_{-i})$$

Definition 2 *We say that q is a communication equilibrium of the FPA if for all $i \in \{1,2\}, v_i, \bar{v}_i \in \{0,1\}$ and for all measurable r we have that:*

$$U_i(v_i, \bar{v}_i, r) \leq U_i(v_i, v_i, id)$$

where id is the identity mapping. That is, sincereness and obedience is a Bayes-Nash equilibrium of the extended FPA.

6.2 Theorem on the Impossibility of Credible Communication

Lemma 1 *In any communication equilibrium a bidder with type $v_i = 0$ must receive $b_i = 0$ with probability 1.*

Proof. See Lemma A.1. in Fang and Morris (2006). ■

From the above lemma it follows that any communication equilibrium must have the following structure. (For simplicity we concentrate only on communication devices which handle the bidders symmetrically.) Let $F : [0,1] \to [0,1]$ be a cumulative distribution function and let $q(1,0)(.,0) = q(0,1)(0,.)$ be its corresponding Borel measure, and set $q(0,0)$ to be the Dirac measure corresponding to the bid profile $(0,0)$. That is, $q(1,0)(B)$ can be positive only if there is $(b,c) \in B$ such that $c = 0$. Similarly, $q(0,1)(B)$ can be positive only if there is $(b,c) \in B$ such that $b = 0$. The positive measures are determined by F. Otherwise sets have 0 measure under $q(1,0)$ and $q(0,1)$. Thus we only have to define $q(1,1)$ and specify F. Now we turn to those canonical communication devices which correspond to the publicly or privately and conditionally independently signalling center. Some more notation is needed. Let $X, (Y)$ be continuous cumulative distribution functions with $X' = x, Y' = y$ and with the interpretation that the random signal is selected according to x (y) when a bidder has reported that he is of low (high) type, that is, 0 (1). Assume that x/y is non increasing. Now let $t_1 : [0,1]^2 \to \Delta[0,1]$ with the interpretation that $t_1(s)$ is the mixed strategy that a high type bidder 1 follows in equilibrium after receiving the signal s. Let $t_2(s_1, s_2) = t_1(s_2, s_1)$ for all $s = (s_1, s_2) \in [0,1] \times [0,1]$ and $T_1(s), T_2(s)$ be the corresponding cumulative distribution functions and define $q(1,1)$ as the Borel measure corresponding to $G(b,c) = \int\int T_1(s_1,s_2)(b)T_2(s_1,s_2)(c)dYdY$ and set $F(b) = \int\int T_1(s_1,s_2)(b)dXdY$.

We say that q is a *canonical private conditionally independent communication equilibrium* if t_1 depends only on its first coordinate and q is a communication equilibrium.

We say that q is a *canonical public conditionally independent communication equilibrium* if T_1, T_2 is an equilibrium of the FPA with public signal structure (x,y) and q is a communication equilibrium.

Theorem 3 *There exists no canonical public or private conditionally independent communication equilibrium of the FPA, where bidder's payoff is higher than in the unique Bayes-Nash equilibrium of the FPA.*

The Theorem has an obvious corollary which states that the bidders cannot collude in equilibrium using plain, unmediated cheap talk messages. To state the corollary formally, we say that FPA is extended with possibly several stages of cheap talk if bidders before submitting bids in the FPA have the possibility to send (possibly simultaneously) messages to each other from a message set M. Histories and strategy sets are defined in the natural way.

Corollary 1 *There is no equilibrium in the cheap talk extension of the FPA, where bidder's payoff is higher than in the unique Bayes-Nash equilibrium of the FPA.*

Proof. Clearly, if there was one collusive equilibrium of the cheap talk extension, then there would be a canonical public conditionally independent communication equilibrium by the revelation principle, which also improves on the FPA's unique equilibrium. ■

7 Discussion

7.1 Conjectures

Let us list some conjectures which we were not able to prove.

We have not been able to fully characterize the optimal signal structure for any given $n > 2$ yet. Nevertheless, we have the following conjectures:

Conjecture 1 *For any $n \geq 2$ the optimal private or public signal structure $(x, y)_n^\star$ satisfies the followings: $0 < y_1 = \ldots = y_{n-1} < y_n$ and $x_1 > \ldots > x_n = 0$.*

Conjecture 2 *For any $n \geq 2$ the optimal private structure is strictly better for the bidders than the optimal public signal structure.*

This result could have important implications about the way the information is shared by trade associations. That is, usually a trade association provides the same information to all its members, while the above conjecture indicates that the members' payoffs could be increased by providing differentiated information. Though, the dissemination of differentiated information also requires from the members a higher trust in the trade association, which may not be possible in practice.

Conjecture 3 *If there are infinitely many signals no optimal structure exists, and payoffs are converging to a supremum P.*

An interesting question concerns the magnitude of this limit P.

Conjecture 4 *The correlated signal structure demonstrated in the next subsection gives strictly better payoffs than P.*

An open question is, whether one can construct a collusive mechanism with correlated signals in which the center does not know bidders' valuations and can impose no constraint on their bids. Our conjecture is:

Conjecture 5 *There exists no communication equilibrium in which bidders expect a higher payoff than in the Bayes-Nash equilibrium of the FPA.*

The conjecture is in line with our findings. The closest result we are aware of trying to prove the conjecture is Lopomo, Marx, and Sun (2009). They use linear programming technique to show this impossibility. To be able to apply LP, they assume that bidders place their bids on a grid, and show that as the grid becomes finer the best collusive payoff converges to the non-cooperative payoff level. Pavlov (2009) proves our conjecture for all pay auctions. He also shows that, in a general FPA (with large type space) as the number of bidders increases the best collusive outcome converges to the non-cooperative one.

7.2 Correlation

We have considered the case when the signals are generated independently. The following example demonstrates that we can improve bidders' payoffs even further by introducing correlation between the private signals. We conjecture that the equilibrium below is better for the bidders than the equilibrium with independent signals for any n.

Example 4 *Let $n = 2$ and let the joint distribution of types be*

	0	1	2
0	p^2	$p(1-p)$	0
1	$p(1-p)$	0	$\frac{(1-p)^2}{2}$
2	0	$\frac{(1-p)^2}{2}$	0

We claim that a bidder of type 1 will randomize according to

$$G_1(b) = \frac{2p}{1+p}\frac{1}{1-b}$$

on the interval $\left[0, \frac{1-p}{1+p}\right]$, *while a bidder of type* 2 *will randomize according to*

$$G_2(b) = \frac{2p}{1-p}\frac{b}{1-b}$$

on the interval $\left(0, \frac{1-p}{1+p}\right]$. *Suppose bidder 2 follows this strategy. Consider bidder 1 of type* 1. *Her expected payoff is*

$$\frac{p(1-p) + \frac{(1-p)^2}{2}G_2(b)}{p(1-p) + \frac{(1-p)^2}{2}}(1-b) = \frac{2p}{1+p}.$$

Consider bidder 1 of type 2. *Her expected payoff is*

$$G_1(b)(1-b) = \frac{2p}{1+p}.$$

Thus, it is also optimal for bidder 1 to follow the given strategy.

When $p = 0.25$, *the bidder's ex ante payoff is* 0.3, *while the seller's revenue is* 0.3375.

7.3 Adverse selection

McAfee and McMillan (1992) investigates optimal collusion in the presence of "coordinative mechanisms" where bidders report their types to the center and the center submits the corresponding bids. In the first price auction with a reserve price $r \geq 0$, under the optimal collusion scheme, bidders truthfully report their valuations, and the center bids r for the bidders whose valuation is not less than r. That is, a center, who does not know the valuations but has the right to submit bids, solves the adverse selection problem, meanwhile abstracting away from the moral hazard problem.

A center which submits the bids can be thought of as a constraint on the bidders' admissible bids, which may depend on the reports. Let us introduce now a center which faces both the adverse selection and the moral hazard problem. We let the center to impose a less stringent constraint compared to that of McAfee and McMillan (1992). The constraint is as follows: a bidder who reported v' about her valuation is not allowed to bid more than v'.[19] Suppose now that bidders are asked about their valuations and bid

[19] Such a center can be thought of as a regulator which asks the participants to report about their costs c. If the report is c', the regulator does not allow to bid $b < c'$ in a procurement auction. Participants may also get information about each others' costs through the regulator.

according to the signals they have received from the center. The only possible profitable deviation is to report valuation V_l when the bidder is of valuation type V_h and then bid above V_l. As a consequence, if the center can impose the constraint that bids have to be lower than the reports, such a deviation will not be profitable, which solves the adverse selection problem from which we abstracted away at the beginning.

Notice also this constraint disciplines only the reports. Once the reports are truthful, the constraint obviously does not bind. Moreover, this observation is not a special feature of our model, but the same is true in the multidimensional type setup of Bergemann and Vällimäki (2006) and Kim and Che (2004).

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8 Appendix

Proof of Proposition 1. First, note that $F_j\left(\bar{b}_j\right) = 1$ for all $j \in n$ is satisfied and so the mixed strategies are well defined. Suppose that bidder 2 follows the strategy given in Proposition 1 and $y_1 > 0$. Consider bidder 1 of type $(1, j), j \in n$. Her expected payoff when bidding $b \in \left[\bar{b}_{j-1}, \bar{b}_j\right]$ is

$$\left\{\frac{px_j + (1-p)\, y_j \sum_{k=1}^{j-1} y_k}{px_j + (1-p)\, y_j} + \frac{(1-p)\, y_j^2}{px_j + (1-p)\, y_j} F_j\left(b\right)\right\}\left(1-b\right).$$

Substituting $F_j(b)$ from (1) yields a positive constant

$$\frac{px_j + (1-p)\, y_j \sum_{k=1}^{j-1} y_k}{px_j + (1-p)\, y_j} \left(1 - \bar{b}_{j-1}\right).$$

Therefore, bidder 1 is indeed indifferent between any bid in the interval $\left[\bar{b}_{j-1}, \bar{b}_j\right]$.

Suppose now that bidder 1 of type $(1, j)$ bids in an interval $\left[\bar{b}_{l-1}, \bar{b}_l\right]$ for $l \neq j$. Her expected payoff is

$$\begin{aligned}
& \left\{\frac{px_j + (1-p)\, y_j \sum_{k=1}^{l-1} y_k}{px_j + (1-p)\, y_j} + \frac{(1-p)\, y_j y_l}{px_j + (1-p)\, y_j} F_l(b)\right\}(1-b) \\
= \;& \frac{px_j + (1-p)\, y_j \sum_{k=1}^{l-1} y_k}{px_j + (1-p)\, y_j}(1-b) \\
& + \frac{(1-p)\, y_j y_l}{px_j + (1-p)\, y_j} \frac{px_l + (1-p)\, y_l \sum_{k=1}^{l-1} y_k}{(1-p)\, y_l^2}\left(b - \bar{b}_{l-1}\right) \\
= \;& \frac{b}{px_j + (1-p)\, y_j}\left\{\frac{y_j}{y_l}\left(px_l + (1-p)\, y_l \sum_{k=1}^{l-1} y_k\right) - \left(px_j + (1-p)\, y_j \sum_{k=1}^{l-1} y_k\right)\right\} + \Phi \\
= \;& \frac{pb}{px_j + (1-p)\, y_j}\left(\frac{y_j}{y_l} x_l - x_j\right) + \Phi,
\end{aligned}$$

where the rest of the terms that do not contain b are collected in the parameter Φ. Since $x_l/y_l > x_j/y_j$ for all $l < j$, it follows that the payoff of type $(1, j)$ is increasing in b for $b < \bar{b}_{j-1}$ and therefore bidder 1 of type $(1, j)$ does not want to deviate by bidding below $\bar{b}_{j-1}$. Similarly, since $x_l/y_l < x_j/y_j$ for all $l > j$, it follows that the payoff of type $(1, j)$ is decreasing in b for $b > \bar{b}_j$ and therefore bidder 1 of type $(1, j)$ does not want to deviate by bidding above $\bar{b}_j$ either. If $y_1 = 0$, using the tie breaking rule, the proof is basically the same. Finally, bidder 1 of type $(0, j)$ will not bid above zero as it implies a negative expected payoff, while any lower bid would still give a payoff of 0.[20] ■

Proof of Theorem 1. Using (2) and (3), the expected payoff of a bidder can be expressed as

$$P(x, y) = (1-p)\left(z_1 + z_1 z_2 + z_1 z_2 z_3 + \ldots + z_1 z_2 \ldots z_{n-1} z_n\right),$$

[20] It appears that one can also demonstrate the uniqueness of this equilibrium along the lines of Fang and Morris (2006).

where

$$z_i = \frac{px_i + (1-p)\, y_i \sum_{j=1}^{i-1} y_j}{px_{i-1} + (1-p)\, y_{i-1} \sum_{j=1}^{i-1} y_j}$$

for $i > 1$ and $z_1 = px_1$. The proof consists of two parts. First, we show that a signal structure $(x, y)_n$ with $x_n > 0$ cannot be optimal. Next, we show for any signal structure $(x, y)_n$ with $x_n = 0$ how we can increase bidder's payoff by introducing an additional signal.

Assume first that $x_n > 0$. Let us define a new signal structure $(x', y')_n$ as follows: $x'_i = x_i$ for $i = 1, ..., n-2$, $x'_{n-1} = x_{n-1} + \epsilon$, $x'_n = x_n - \epsilon \geq 0$, and $y'_i = y_i$ for $i = 1, ..., n$. ϵ must be sufficiently small to ensure that $x'_{n-2}/y'_{n-2} > x'_{n-1}/y'_{n-1}$ holds.[21] Given the assumptions about $(x', y')_{n+1}$, we only need to show that

$$z_{n-1}\,(1 + z_n) < z'_{n-1}\,(1 + z'_n)$$

or

$$\frac{px_{n-1} + (1-p)\, y_{n-1} \sum y_j}{px_{n-2} + (1-p)\, y_{n-2} \sum y_j} \times \left(1 + \frac{px_n + (1-p)\, y_n \left(\sum y_j + y_{n-1}\right)}{px_{n-1} + (1-p)\, y_{n-1} \left(\sum y_j + y_{n-1}\right)}\right) <$$
$$\frac{p\,(x_{n-1} + \epsilon) + (1-p)\, y_{n-1} \sum y_j}{px_{n-2} + (1-p)\, y_{n-2} \sum y_j} \times \left(1 + \frac{p\,(x_n - \epsilon) + (1-p)\, y_n \left(\sum y_j + y_{n-1}\right)}{p\,(x_{n-1} + \epsilon) + (1-p)\, y_{n-1} \left(\sum y_j + y_{n-1}\right)}\right)$$

where we write $\sum y_j$ instead of $\sum_{j=1}^{n-2} y_j$. The above expression is equivalent to

$$\frac{px_{n-1} + (1-p)\, y_{n-1} \sum y_j}{px_{n-1} + (1-p)\, y_{n-1} \left(\sum y_j + y_{n-1}\right)} < \frac{p\,(x_{n-1} + \epsilon) + (1-p)\, y_{n-1} \sum y_j}{p\,(x_{n-1} + \epsilon) + (1-p)\, y_{n-1} \left(\sum y_j + y_{n-1}\right)}.$$

The right hand side is increasing in ϵ and since both sides are equivalent for $\epsilon = 0$, it follows that we can always increase the payoff by shifting some probability away from x_n to x_{n-1}. Thus, $x_n > 0$ cannot hold in the optimum.

Suppose now that $x_n = 0$. We introduce an additional signal in the following way: $x'_i = x_i$ and $y'_i = y_i$ for $i = 1, ..., n-2$, $x'_{n-1} + x'_n = x_{n-1}$ and $y'_{n-1} + y'_n + y'_{n+1} = y_{n-1} + y_n$, and let $x'_{n+1} = 0$. Now we need to compare $z_{n-1}\,(1 + z_n)$ with $z'_{n-1}\left(1 + z'_n\left(1 + z'_{n+1}\right)\right)$.

[21] It can also be shown that it is always possible to increase bidder's payoff by introducing an extra signal when $x_n > 0$. It can be done by defining the new probabilities in the following way: $x'_i = x_i$ and $y'_i = y_i$ for $i = 1, ..., n-1$, and $x'_n = x_n/2 + \epsilon$, $x'_{n+1} = x_n/2 - \epsilon \geq 0$, $y'_n = y_n/2$, and $y'_{n+1} = y_n/2$, where ϵ is such that $x'_{n-1}/y'_{n-1} > x'_n/y'_n$ holds.

Now

$$z_{n-1}\left(1+z_{n}\right)=\frac{px_{n-1}+\left(1-p\right)y_{n-1}\sum y_{j}}{px_{n-2}+\left(1-p\right)y_{n-2}\sum y_{j}}\left(1+\frac{\left(1-p\right)y_{n}\left(\sum y_{j}+y_{n-1}\right)}{px_{n-1}+\left(1-p\right)y_{n-1}\left(\sum y_{j}+y_{n-1}\right)}\right)$$

where we write $\sum y_j$ instead of $\sum_{j=1}^{n-2} y_j$, while

$$\begin{aligned}z'_{n-1}\left(1+z'_{n}\left(1+z'_{n+1}\right)\right) &= \frac{px'_{n-1}+\left(1-p\right)y'_{n-1}\sum y_{j}}{px_{n-2}+\left(1-p\right)y_{n-2}\sum y_{j}}\times\\ &\times\left(1+\frac{px'_{n}+\left(1-p\right)y'_{n}\left(\sum y_{j}+y'_{n-1}\right)}{px'_{n-1}+\left(1-p\right)y'_{n-1}\left(\sum y_{j}+y'_{n-1}\right)}\times\right.\\ &\times\left.\left(1+\frac{\left(1-p\right)y'_{n+1}\left(\sum y_{j}+y'_{n-1}+y'_{n}\right)}{px'_{n}+\left(1-p\right)y'_{n}\left(\sum y_{j}+y'_{n-1}+y'_{n}\right)}\right)\right).\end{aligned}$$

Desired inequality

$$z_{n-1}\left(1+z_{n}\right)<z'_{n-1}\left(1+z'_{n}\left(1+z'_{n+1}\right)\right)$$

is satisfied if

$$\begin{aligned}&\frac{px_{n-1}+\left(1-p\right)y_{n-1}\sum y_{j}}{px'_{n-1}+\left(1-p\right)y'_{n-1}\sum y_{j}}\left(1+\frac{\left(1-p\right)y_{n}\left(\sum y_{j}+y_{n-1}\right)}{px_{n-1}+\left(1-p\right)y_{n-1}\left(\sum y_{j}+y_{n-1}\right)}\right)\\ <\;& 1+\frac{px'_{n}+\left(1-p\right)y'_{n}\left(\sum y_{j}+y'_{n-1}\right)}{px'_{n-1}+\left(1-p\right)y'_{n-1}\left(\sum y_{j}+y'_{n-1}\right)}\left(1+\frac{\left(1-p\right)y'_{n+1}\left(\sum y_{j}+y'_{n-1}+y'_{n}\right)}{px'_{n}+\left(1-p\right)y'_{n}\left(\sum y_{j}+y'_{n-1}+y'_{n}\right)}\right)\end{aligned}$$

or

$$\begin{aligned}&\frac{px_{n-1}+\left(1-p\right)y_{n-1}\sum y_{j}}{px'_{n-1}+\left(1-p\right)y'_{n-1}\sum y_{j}}\times\\ &\times\frac{px_{n-1}+\left(1-p\right)\left(y_{n-1}+y_{n}\right)\left(\sum y_{j}+y_{n-1}\right)}{px_{n-1}+\left(1-p\right)y_{n-1}\left(\sum y_{j}+y_{n-1}\right)}-1\\ <\;&\frac{px'_{n}+\left(1-p\right)y'_{n}\left(\sum y_{j}+y'_{n-1}\right)}{px'_{n-1}+\left(1-p\right)y'_{n-1}\left(\sum y_{j}+y'_{n-1}\right)}\times\\ &\times\frac{px'_{n}+\left(1-p\right)\left(y'_{n}+y'_{n+1}\right)\left(\sum y_{j}+y'_{n-1}+y'_{n}\right)}{px'_{n}+\left(1-p\right)y'_{n}\left(\sum y_{j}+y'_{n-1}+y'_{n}\right)}.\end{aligned} \tag{8}$$

Let us define the new probabilities as follows:

$$\begin{aligned} x'_{n-1} &= (1-k)\,x_{n-1}, \\ x'_n &= kx_{n-1}, \\ x'_{n+1} &= 0, \\ y'_{n-1} &= (1-k)\,y_{n-1}, \\ y'_n &= ky_{n-1}+\epsilon, \\ y'_{n+1} &= y_n-\epsilon, \end{aligned}$$

where $k \in (0,1)$ and $\epsilon \in (0, y_n)$.

Using that

$$\frac{x_{n-1}}{y_{n-1}} = \frac{x'_{n-1}}{y'_{n-1}},$$

the left hand side of (8) becomes

$$\frac{px_{n-1} + (1-p)\,(y_{n-1}+y_n)\left(\sum y_j + y_{n-1}\right)}{px'_{n-1} + (1-p)\,y'_{n-1}\left(\sum y_j + y_{n-1}\right)} - 1.$$

Further, using that $x_{n-1} = x'_{n-1} + x'_n$ and $y_{n-1} + y_n = y'_{n-1} + y'_n + y'_{n+1}$, it can be written as

$$\frac{px'_n + (1-p)\left(y'_n + y'_{n+1}\right)\left(\sum y_j + y_{n-1}\right)}{px'_{n-1} + (1-p)\,y'_{n-1}\left(\sum y_j + y_{n-1}\right)}.$$

Thus, it remains to check whether the following inequality

$$\begin{aligned} &\frac{px'_n + (1-p)\left(y'_n + y'_{n+1}\right)\left(\sum y_j + y_{n-1}\right)}{px'_{n-1} + (1-p)\,y'_{n-1}\left(\sum y_j + y_{n-1}\right)} \\ < \; &\frac{px'_n + (1-p)\,y'_n\left(\sum y_j + y'_{n-1}\right)}{px'_{n-1} + (1-p)\,y'_{n-1}\left(\sum y_j + y'_{n-1}\right)} \times \\ &\times \frac{px'_n + (1-p)\left(y'_n + y'_{n+1}\right)\left(\sum y_j + y'_{n-1} + y'_n\right)}{px'_n + (1-p)\,y'_n\left(\sum y_j + y'_{n-1} + y'_n\right)} \end{aligned}$$

or, after substituting for x'_{n-1}, x'_n, y'_{n-1}, and y'_n, and re-arranging, whether the following inequality

$$
\begin{aligned}
&\frac{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+(1-k)\,y_{n-1}\right)}\times \\
&\times\frac{pkx_{n-1}+(1-p)\,(ky_{n-1}+\epsilon)\left(\sum y_j+(1-k)\,y_{n-1}\right)}{pkx_{n-1}+(1-p)\,(ky_{n-1}+\epsilon)\left(\sum y_j+y_{n-1}+\epsilon\right)} \\
&-\frac{pkx_{n-1}+(1-p)\,(ky_{n-1}+y_n)\left(\sum y_j+y_{n-1}\right)}{pkx_{n-1}+(1-p)\,(ky_{n-1}+y_n)\left(\sum y_j+y_{n-1}+\epsilon\right)} \quad > \quad 0
\end{aligned}
$$

is true.

We take the first order Taylor expansion of the above expression at $\epsilon = 0$ to see if for $\epsilon > 0$ this expression is strictly positive. If the above expression is represented as $f(\epsilon)$, then $f(\epsilon) = f(0) + f'(0)\,\epsilon + R(\epsilon)$. Notice that if $\epsilon = 0$, then the above expression is equal to 0, that is, $f(0) = 0$, while $f'(0)$ is

$$
\begin{aligned}
&\frac{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+(1-k)\,y_{n-1}\right)} \\
&\times\left\{\frac{(1-p)\left(\sum y_j+(1-k)\,y_{n-1}\right)}{pkx_{n-1}+(1-p)\,ky_{n-1}\left(\sum y_j+y_{n-1}\right)}\right. \\
&-\frac{pkx_{n-1}+(1-p)\,ky_{n-1}\left(\sum y_j+(1-k)\,y_{n-1}\right)}{pkx_{n-1}+(1-p)\,ky_{n-1}\left(\sum y_j+y_{n-1}\right)}\times \\
&\left.\times\frac{(1-p)\left(\sum y_j+(1+k)\,y_{n-1}\right)}{pkx_{n-1}+(1-p)\,ky_{n-1}\left(\sum y_j+y_{n-1}\right)}\right\} \\
&+\frac{pkx_{n-1}+(1-p)\,(ky_{n-1}+y_n)\left(\sum y_j+y_{n-1}\right)}{pkx_{n-1}+(1-p)\,(ky_{n-1}+y_n)\left(\sum y_j+y_{n-1}\right)}\times \\
&\times\frac{(1-p)\,(ky_{n-1}+y_n)}{pkx_{n-1}+(1-p)\,(ky_{n-1}+y_n)\left(\sum y_j+y_{n-1}\right)},
\end{aligned}
$$

which can be simplified to

$$
\begin{aligned}
&\frac{1}{k}\times\frac{(1-p)\left(\sum y_j+(1-k)\,y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+(1-k)\,y_{n-1}\right)} \\
&-\frac{1}{k}\times\frac{(1-p)\left(\sum y_j+(1+k)\,y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)} \\
&+\frac{(1-p)\,(ky_{n-1}+y_n)}{pkx_{n-1}+(1-p)\,(ky_{n-1}+y_n)\left(\sum y_j+y_{n-1}\right)}.
\end{aligned}
$$

To simplify notation, let $\Psi = \frac{p}{1-p} x_{n-1}$. Then

$$\begin{aligned}
& \frac{1}{k} \times \frac{\sum y_j + (1-k)\, y_{n-1}}{\Psi + y_{n-1}\left(\sum y_j + (1-k)\, y_{n-1}\right)} \\
& -\frac{1}{k} \times \frac{\sum y_j + (1+k)\, y_{n-1}}{\Psi + y_{n-1}\left(\sum y_j + y_{n-1}\right)} + \frac{k y_{n-1} + y_n}{k\Psi + (k y_{n-1} + y_n)\left(\sum y_j + y_{n-1}\right)} \\
= \; & \frac{\Psi}{\left(k\Psi + (k y_{n-1} + y_n)\left(\sum y_j + y_{n-1}\right)\right)} \times \\
& \times \frac{\Psi y_n - k y_{n-1}\left(\Psi + y_{n-1}\right)}{\left(\Psi + y_{n-1}\left(\sum y_j + (1-k)\, y_{n-1}\right)\right)\left(\Psi + y_{n-1}\left(\sum y_j + y_{n-1}\right)\right)}.
\end{aligned}$$

It follows that if k is chosen such that

$$0 < k < \min\left(1, \frac{\Psi y_n}{y_{n-1}\left(\Psi + y_{n-1}\right)}\right), \tag{9}$$

then the derivative $f'(0)$ is positive, which was necessary to prove.

Finally,

$$R(\epsilon) = \frac{f''(\theta)}{2}\epsilon^2$$

where $\theta \in [0, \epsilon]$. $f''(\theta)$ exists and is finite for $\theta \in [0, \epsilon]$. Therefore, for a given k, satisfying (9), we can always select ϵ satisfying

$$\left(f'(0) + \frac{f''(\theta)}{2}\epsilon\right)\epsilon > 0.$$

■

Proof of Proposition 2. First, note that $F_{lk}\left(\bar{b}_k\right) = F_{kl}\left(\bar{b}_k\right) = 1$ is satisfied and so the mixed strategies are well defined. The expected payoff of bidder 1 with valuation $v_1 = 1$ is given by

$$\left\{\frac{p x_k}{p x_k + (1-p)\, y_k} + \frac{(1-p)\, y_k}{p x_k + (1-p)\, y_k} F_{kl}(b)\right\}(1-b). \tag{10}$$

Substituting (5) into (10), we can verify that bidder 1 is indifferent among all bids in the interval $\left[0, \bar{b}_k\right]$, earning an expected payoff of π_k given in (6). Similarly, the expected payoff of bidder 2 with valuation $v_2 = 1$ is

$$\left\{\frac{p x_l}{p x_l + (1-p)\, y_l} + \frac{(1-p)\, y_l}{p x_l + (1-p)\, y_l} F_{lk}(b)\right\}(1-b). \tag{11}$$

Substituting (4) into (11), we can verify that bidder 2 is also indifferent among all bids in the interval $[0, \bar{b}_k]$, earning the same expected payoff π_k. Obviously, no bidder has incentives to bid above $\bar{b}_k$, while any bid below 0 would give a payoff of 0. Thus, we can conclude that (4) and (5) represent equilibrium strategies of high valuation bidders when signals are k and l, with $k \leq l$. Finally, a bidder with $v = 0$ will not bid above zero as it implies a negative expected payoff, while any lower bid would still give a payoff of 0. ■

Proof of Theorem 2. Consider introducing signal 0 and re-assign the probabilities as follows:

$$\begin{aligned} x_0' &= x_1/2 + \epsilon, \\ x_1' &= x_1/2 - \epsilon, \\ y_0' &= y_1/2, \\ y_1' &= y_1/2, \end{aligned}$$

and $x_j' = x_j$ and $y_j' = y_j$ for $j = 2, ..., n$, with $\epsilon > 0$. Note that ϵ must be sufficiently small to ensure that

$$\frac{x_1'}{y_1'} > \frac{x_2'}{y_2'}.$$

Also, note that the construction requires that $y_1 > 0$. It will be argued later that it is not optimal to have $y_1 = 0$ under the optimal signal structure.

Let $\tilde{\Pi}$ denote a matrix that is obtained from Π by eliminating the first column and row. Similarly, let $\tilde{u}$ and $\tilde{y}$ denote vectors obtained from u and y, respectively, by eliminating the first element. Then the bidder's payoff, conditional on having the high valuation, for n signal case can be written as

$$\frac{R(x, y)}{1 - p} = px_1 \left(1 - y_1 + \frac{y_1}{px_1 + (1 - p) y_1} \right) + \tilde{u}' \tilde{\Pi} \tilde{y}.$$

In a similar manner, the bidder's payoff after introducing signal 0 can be written as

$$\begin{aligned}\frac{R(x', y')}{1-p} &= px_0'\left(1 - y_0' + \frac{y_0'}{px_0' + (1-p)y_0'}\right) \\ &\quad + px_1'\left(1 - y_1' + \frac{y_1'}{px_1' + (1-p)y_1'}\right) \\ &\quad - px_1'y_0' - \frac{px_0' + (1-p)y_0'}{px_1' + (1-p)y_1'}px_1'y_1' \\ &\quad + \tilde{u}'\tilde{\Pi}\tilde{y}.\end{aligned}$$

After some manipulation, to show that $R(x', y') > R(x, y)$ is equivalent to showing that

$$\begin{aligned}&\frac{x_0'y_0'}{px_0' + (1-p)y_0'} + \frac{x_1'y_1'}{px_1' + (1-p)y_1'} + x_0'y_1' - \frac{px_0' + (1-p)y_0'}{px_1' + (1-p)y_1'}x_1'y_1' \\ > \; &\frac{x_1y_1}{px_1 + (1-p)y_1}\end{aligned}$$

or

$$\begin{aligned}&\frac{(x_1 + 2\epsilon)}{p(x_1 + 2\epsilon) + (1-p)y_1} + \frac{(x_1 - 2\epsilon)}{p(x_1 - 2\epsilon) + (1-p)y_1} \\ &+ (x_1/2 + \epsilon) - \frac{p(x_1 + 2\epsilon) + (1-p)y_1}{p(x_1 - 2\epsilon) + (1-p)y_1}(x_1/2 - \epsilon) \\ > \; &\frac{2x_1}{px_1 + (1-p)y_1}.\end{aligned}$$

Denote the left hand side of the above expression as $f(\epsilon)$. Then $f(\epsilon) = f(0) + f'(0)\epsilon + R(\epsilon)$. Note that $f(0) = 2x_1/(px_1 + (1-p)y_1)$, therefore we only need to show that $f'(0) > 0$ to establish the desired inequality. $f'(0)$ is given by

$$2\left(1 - \frac{px_1}{px_1 + (1-p)y_1}\right) > 0.$$

Thus, adding an extra signal increases the expected payoff of bidder.

To complete the proof, we show that it is not optimal to set $y_1 = 0$. Take a signal structure (x, y) such that $y_1 = 0$. Consider taking a small probability ϵ away from y_2 and assigning it to y_1, without changing the rest of probabilities. This only affects the first two columns and rows of matrix Π, and the first two elements of u and y. Writing out the corresponding terms in $R(x, y)$ (and ignoring the rest of the term that do not change), we obtain

$$
\begin{aligned}
& px_1 \left(1 - \epsilon + \frac{\epsilon}{px_1 + (1-p)\,\epsilon}\right) \\
& + px_2 \left(1 - y_2 + \epsilon + \frac{y_2 - \epsilon}{px_2 + (1-p)(y_2 - \epsilon)}\right) \\
& - px_2 \epsilon - \frac{px_1 + (1-p)\,\epsilon}{px_2 + (1-p)(y_2 - \epsilon)} px_2 (y_2 - \epsilon) .
\end{aligned}
$$

Rearranging, gives

$$
p \left(\frac{x_1 \epsilon}{px_1 + (1-p)\,\epsilon} + \frac{x_2 (y_2 - \epsilon)}{px_2 + (1-p)(y_2 - \epsilon)} \right) (1 - px_1 - (1-p)\,\epsilon) + \Phi,
$$

where Φ contains terms independent of ϵ.

We want to know if the derivative of the above expression with respect to ϵ is positive at $\epsilon = 0$. If so, it would imply that the bidder's payoff can be increased by shifting some probability from y_2 to y_1. The derivative is

$$
\left(1 - \left(\frac{px_2}{px_2 + (1-p)\,y_2}\right)^2\right)(1 - px_1) - \frac{p(1-p)\,x_2 y_2}{px_2 + (1-p)\,y_2},
$$

To verify that the derivative is positive, multiply it with $(px_2 + (1-p)\,y_2)^2$ and rearrange to obtain

$$
(1-p)\,y_2 \left\{ px_2 (1 - px_1) + (px_2 + (1-p)\,y_2)(1 - p(x_1 + x_2)) \right\},
$$

which is indeed positive. ■

Proof of Theorem 3.

Case 1 (No Private Communication Equilibrium) *Let $p = 0.5$ for simplicity. The result naturally extends to other values of p. Let $q : \{0,1\}^2 \to \Delta[0,1]^2$ a canonical communication device defined above. Denote with $g(b) = G'(b,1)$ and $f(b) = F'(b)$, whenever F, G are differentiable (almost everywhere). In case of countably many signals it can be easily seen from 1 that $f/g = x_i/y_i$ is a step function. Suppose q is a communication equilibrium. Then the followings should hold: $\forall b \in supp\ g \cup supp\ f$*

$$\frac{f(b) + g(b)G(b)}{f(b) + g(b)}(1-b) \geq \frac{f(b) + g(b)G(c)}{f(b) + g(b)}(1-c) \tag{12}$$

This means that assuming sincere report and by independent recommendations, after recommendation b, b must maximize the RHS of inequality (12), the moral hazard condition. The necessary conditions are FOC:

$$f(b) = g(b)(g(b)(1-b) - G(b)) \tag{13}$$

and SOC:

$$\frac{\partial \frac{f(b)}{g(b)}}{\partial b} \leq 0. \tag{14}$$

(14) means a non-increasing likelihood ratio from which it follows that $F(b) \geq G(b)$. That is, the equilibrium conditions simply imply the monotone likelihood ratio. Moreover, (13) shows that g uniquely determines f and the high type's expected payoff in equilibrium is

$$0.5 \int_0^1 (f(b) + g(b)G(b))(1-b)db = 0.5 \int_0^1 (1-b)^2 g^2(b)db. \tag{15}$$

Assume that (15) is larger than 0.5 since we assume the existence of a collusive communication equilibrium. Now we wish to prove that, if (15) is larger than 0.5 then

$$\int_0^1 (f(b) + g(b)G(b))(1-b)db < \max_b (1 + F(b))(1-b) \tag{16}$$

That is, we want to show in (16) that high type bidders are better off when reporting 0 to q and choosing their bids optimally.

We show there is a c which maximizes $F(b) - G(b)$ *and* $(1 + G(b))(1 - b)$ *at the same time. A high type bidder, who reports 0 and bids c will be better off then the LHS of (16).*

If c maximizes $F(b)-G(b)$ *then it must be that* $F(b)-G(b) \leq F(c)-G(c)$ *for all b. If* $c > b$*, then* $G(c) - G(b) \leq F(c) - F(b)$ *and writing* $\int_b^c g(a)da \leq \int_b^c f(a)da$ *we have that it is true if* $f(a) \geq g(a)$*. If* $c < b$*, then* $G(b)-G(c) \geq F(b) - F(c)$ *and writing* $\int_c^b g(a)da \geq \int_c^b f(a)da$*, we have that it is true if* $f(a) \leq g(a)$*.*

Changing the roles of b and c in (12) and rearranging, we have that:

$$(1-b) + \frac{f(c)}{g(c)}(b-c) + G(c)(1-c) \geq G(b)(1-b) + (1-b)$$

Now, if $b > c$ *and* $\frac{f(c)}{g(c)} \leq 1$ *or* $b < c$ *and* $\frac{f(c)}{g(c)} \geq 1$*, we can have that:*

$$(1-c) + G(c)(1-c) \geq G(b)(1-b) + (1-b)$$

So we can choose $c = sup\{b : \frac{f(b)}{g(b)} \geq 1\}$*.*

Now we show that reporting 0 and bidding c is a profitable deviation for a high type bidder. The LHS of (16) can be written as

$$\begin{aligned} & \int_0^{b_m} (f(b) - g(b))(1-b)db + \int_0^1 g(b)(1+G(b))(1-b)db \\ = & \int_0^{b_m} F(b) - G(b)db + \int_0^1 (1+G(b))(1-b)g(b)db \\ \leq & (F(c) - G(c))b_m + (1+G(c))(1-c), \end{aligned}$$

where b_m *is the highest possible bid. Evaluating the RHS of (16) at* $b = c$ *and combining it with the LHS of (16) we have that:*

$$(F(c) - G(c))b_m < (F(c) - G(c))(1-c)$$

if $b_m < (1-c)$*.*

Finally we argue by reaching a contradiction that $1-c \leq b_m$ *is impossible. We show that a sincere high type must expect a payoff which is less than 0.5 if* $1 - c \leq b_m$*. First, notice that by (13) (FOC):* $g(b)(1-b) - G(b) \geq 0$ *for all b. Rearranging and integrating from* $d > 0$ *to* $b > d$ *we have*

$$\int_d^b \frac{g(t)}{G(t)}dt \geq \int_d^b \frac{1}{1-t}dt$$

or

$$G(b) \geq \frac{1-d}{1-b} G(d).$$

Let $b = b_m$. *Then for all* $d < b_m$ *we must have that* $c \geq 1 - b_m \geq (1-d)G(d)$. *Since* c *maximizes* $(1-b)(1+G(b))$ *it must be that* $1 \geq 1 - c + G(c)(1-c) \geq (1-b)(1+G(b))$ *from which* $G(b) \leq \frac{b}{1-b}$. *Since* $G(0) = 0$ *it must be that* $g(0) \leq 1$ *otherwise the previous inequality would not hold for* b *sufficiently close to 0.*

Finally by (14) (SOC): $g'(b)(1-b) - 2g(b) \leq 0$ *and rearranging and integrating both sides from 0 to* b *we have that:* $(1-b)g(b) - g(0) \leq G(b)$ *from which* $(1-b)g(b) \leq \frac{b}{1-b} + 1 = \frac{1}{1-b}$ *by using the upper bounds on* $g(0)$ *and on* $G(b)$. *That is,* $g(b) \leq \frac{1}{(1-b)^2}$ *and using the RHS of (15) the payoff is smaller than* $0.5 \int_0^{b_m} g(b)db = 0.5$.

Case 2 (No Public Communication Equilibrium) *Assume that* q *is a canonical public conditionally independent communication equilibrium. Take any* $s = (s_1, s_2)$ *and assume without loss of generality that* $x(s_1)/y(s_1) \geq x(s_2)/y(s_2)$. *Then 2 remains true when instead of* $x_{s_1}, y_{s_1}, x_{s_2}, y_{s_2}$ *we write* $x(s_1), y(s_1), x(s_2), y(s_2)$. *Notice that* $T_2(s) = F_s$ *(where* F_s *is* F_{kl} *from 2) and the expected payoff of both bidders when they know the signal* s *is*

$$\pi(s_1) = \frac{px(s_1)}{px(s_1) + (1-p)y(s_1)}$$

which is non-increasing in s_1. *Therefore the expected payoff of an honest high type before receiving the recommendation can be written as:*

$$\Pi = \int_0^1 [(px(s_1) + (1-p)y(s_1))(1 - Y(s_1)) +$$

$$+ y(s_1)(p(1 - X(s_1)) + (1-p)(1 - Y(s_1)))]\pi(s_1)ds_1$$

Now suppose for a moment that the deviator after reporting 0 to q *gets the public signal* s *instead of 0 recommended bid. Then his payoff is*

$$\bar{\Pi} = \int_0^1 [(px(s_1) + (1-p)y(s_1))(1 - X(s_1)) +$$

$$+ x(s_1)(p(1 - X(s_1)) + (1-p)(1 - Y(s_1)))]\pi(s_1)ds_1$$

First we show that $\bar{\Pi} > \Pi$, then we argue that the deviator can get ϵ close to $\bar{\Pi}$ even if he receives only the 0 recommendation from q by simply bidding ϵ. But $\bar{\Pi} > \Pi$ holds since $X > Y$ since x/y is non-increasing (Y first order stochastically dominates X and so $y(1-Y)$ dominates $x(1-X)$) and $\pi(s_1)$ is non-increasing and not constant. Finally, notice that by bidding ϵ the deviator may get less than he gets when he knows the public signal only if b_s (the upper limit of the support of $T(s)$) is smaller than ϵ. In other words, if ϵ is not in the support of $T(s)$. Thus, by bidding ϵ the deviator wins at least $\bar{\Pi} - \epsilon$.

■