Defaulting on sovereign debt: A macroeconomic analysis.

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April 2012

Abstract

We study the sustainability of sovereign debt in a macroeconomic model in which monetary and fiscal policies interact. The inconsistencies between policies may lead to default. We distinguish two critical values: the "default" and "no-default" thresholds. The "default" threshold corresponds to the upper limit for public debt: default intervenes when lenders do not consider that the state is able to fulfill its contractual debt obligation. The "no-default" threshold corresponds to a lower level of public debt. Under this level, and in the absence of future shocks, public debt necessarily converges to its steady state level. Above it, lenders still give a positive probability to the full reimbursement of public debt. However, above threshold and in the absence of future shocks, the risk premium imposed by lenders is such that future default is unescapable. We show that a "successful default" implies a rule of default fulfilling the sustainability criterion of debt after default. Here sustainability means that the risk premium after default must sufficiently small so that the ration of debt to GDP decreases continuously. Such a rule of default implies a sufficiently large reduction in public debt.

JEL Codes : E6; F4

Keywords : Public debt, sovereign default, risk premium, debt sustainability.

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1 Introduction.

The threat of sovereign default sets a dual problem to policymakers. On the one hand, it is important to assess the responsibilities of macroeconomic decisions in a possible default; on the other hand, we need to know what to do after default and which accompanying measures are necessary. It is not disputed that, at least in part, default is due to an erroneous macroeconomic policy strategy, or more precisely, by inconsistent and contradictory monetary and fiscal policies.

The aim of this article is to investigate these two issues using a macroeconomic model which relies on the interaction between possibly inconsistent monetary and fiscal rules, and takes into account the possibility of sovereign default. Default occurs because of a conjunction of reasons: *i)* the existence of a fiscal limit, which forbids an increase in primary deficits, *ii)* an unsatisfactory current or expected macroeconomic situation, *iii)* an "active" monetary policy which does not give up on its inflation stabilization objective despite the prospect of default and *iv)* a high initial public debt level, not too far from the "default" threshold, defined as the upper limit for public debt.

Default occurs when lenders are unwilling to pursue lending to the state, that is, consider that the governement, given existing policies will be unable to fulfill its contractual debt obligation: the debt burden has reached its upper level. Beyond it, the state is forced to default. Within our model, we are able to define this upper limit for the debt to GDP ratio, which we call the "default threshold". We show that it is useful to define another critical value, which we call the "no-default" threshold. This threshold corresponds to a lower value of the debt to GDP ratio. Hence, lenders admit that there is a positive probability that the sovereign debt will be repaid, including interest. However, absent future shocks (positive or negative), if initial debt is above this threshold the dynamics of debt is such that default will necessarily occur in the future. This is due to the snowball effect of risk premium: above this threshold, the risk premium necessary to clear the debt market is too high and triggers the default itself. It is only if debt is below this level that the dynamics of debt converges to its no-default steady state.

This duality helps us to understand the two issues at stake: what are the causes of default? What to do "after default"? In brief, there is default because the dynamics of sovereign debt is unsustainable. Once default occurs, sovereign debt must be rescheduled, in fact reduced: some "rule of default" applies. We leave aside the bargaining of this rule between government and its lenders, which is as we know lengthy, treacherous and obscure. Taking as given a policy rule, we prove that its parameters must be such that rescheduled debt is below the "no-default" threshold. That is, the rescheduling must be "sufficiently large".

We reason on given fiscal and monetary policy rules. Their incompatibility generates default. We do not address the issue of the changing of policy rules which could be decided by policymakers in order to avoid default, precisely because we want to understand circumstances leading to default and what happens after default. The issue of the conflict between monetary and fiscal policies has been addressed by Sargent and Wallace (1981), in a celebrated paper on some "unpleasant monetarist arithmetics", in which they stress that inflation and seignorage is key when the conflict is resolved at the expsense of the monetarist monetary authority. Later Leeper (1991), Sims (1994), Woodford (1994, 1995) address the same issue, developping the "fiscal theory of the price level".[1] However in these studies, sovereign default is ruled out. This is one of the arguments used by Buiter (2002) to criticize the fiscal theory of the price level.

Uribe (2006) centers on the possibility of default in the case when both fiscal and monetary policies are active (in the sense of Leeper) or dominant (in the sense of Sargent and Wallace) and develops a "fiscal theory of default" whereas Blanchard (2004) and Loyo (1999) elaborate on similar grounds a "fiscal theory of inflation".

More recently, Bi (2010), Davig, Leeper and Walker (2011), Daniel and Shiamptanis (2010) and Juessen, Linnemann and Schabert (2011) adress the issue of sovereign default without stressing a macroeconomic view on the unsustainability of public debt.

The paper is organized as follows. Section 2 presents the macroeconomic framework. Section 3 develops an analysis of the various solutions without default. The unsustainability of public debt and default is addressed in Section 4. Section 5 concludes. Proofs are exposed in the Appendix.

2 The economy.

We consider a closed economy with flexible prices and no capital. Money plays no role but prices are expressed in a monetary unit. The role of monetary policy is to stabilize this unit of account over time. Financial markets are complete and public bonds are non-contingent, non-indexed and potentially subject to a risk of default.

[1] See also Cochrane (2001).

2.1 Private sector.

There is a representative agent whose preferences are represented by the following utility function:

$$U_0 = \sum_{t=0}^{+\infty} \beta^t \left[u\left(c_t\right) - v\left(\ell_t\right) \right] \tag{1}$$

with: $u(c_t) = \ln c_t$ and $v(\ell_t) = \delta^{-1} \ell_t^{1+1/\sigma} / (1 + 1/\sigma)$ where ℓ_t represents the supply of labor, c_t, consumption of the good and σ, the Frisch elasticity. These preferences are compatible with growth, and we assume that shocks directly affect the rate of growth of productivity.

At each period, the agent receives labor income, $W_t \ell_t$, and profits Γ_t. She can save, by means of a contingent asset and Treasury bonds of maturity one period. The quantity of issued bonds (held by the agent) at t is denoted by B_t. The amount of redeemed debt is denoted by $h_t B_{t-1}$. h_t denotes the fraction of debt actually reimbursed. It is less than 1 in the case of default.

Income, including financial returns, is taxed at a proportionnal rate τ_t. Denoting by R_t the interest rate offered on bonds and $Q_{t,t+1}$, the price of the contingent asset which generates a nominal return of 1 in any state of nature, the nominal individual budget constraint at t writes:

$$P_t c_t + \frac{B_t}{R_t} + E_t Q_{t,t+1} D_{t+1} \leq (1 - \tau_t) (W_t \ell_t + \Gamma_t) + h_t B_{t-1} + D_t \tag{2}$$

P_t denotes the good price at t. We define the riskless nominal interest rates R_t^f as the inverse comme l'inverse du prix d'un portefeuille certain, *i.e.:*

$$R_t^f = \left(E_t Q_{t,t+1}\right)^{-1}$$

The agent must also satisfy the intertemporal constraint on wealth:

$$h_{t+1} B_t + D_{t+1} \geq -E_{t+1} \sum_{s=t+1}^{\infty} Q_{t+1,s} (1 - \tau_s) (W_s \ell_t + \Gamma_s) \qquad \forall t+1 \tag{3}$$

where D_t denotes the ..., $Q_{t+1,s} \equiv Q_{t+1,t+2} Q_{t+2,t+3} \cdots Q_{s,s}$ and $Q_{t+1,t+1} = 1$.

Maximizing (1) under contraints (2) and (3), the following optimality conditions obtain:

$$\begin{aligned} Q_{t,t+1} &= \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} && (4) \\ R_t^{-1} &= E_t Q_{t,t+1} h_{t+1} && (5) \\ \frac{v'(\ell_t)}{u'(c_t)} &= (1 - \tau_t) \frac{W_t}{P_t} && (6) \end{aligned}$$

and the transversality condition is given by:

$$\lim_{T\to\infty} E_t Q_{t,T}\left[h_T B_T + D_T\right] = 0 \tag{7}$$

The good market is perfectly competitive and returns are constant. The production technology is given by: $y_t \leq A_t \ell_t$, where y_t denotes production and A_t, mean (and marginal) productivity of labor. Profit maximizing lead to standard results on returns: $W_t/P_t = A_t$, $\Gamma_t = 0$ and $y_t = A_t \ell_t$.

2.2 Fiscal and monetary authorities.

Government spends an amount $g_t = \gamma y_t$, and collects taxes on income $\tau_t y_t$. It balances its budget by issuing nominal one-period maturity Treasury bonds at a price $1/R_t$. In case of default at t, it reimburses a fraction $h_t < 1$ of its debt contracted at $t-1$, B_{t-1}. The instantaneous government budget constraint writes:

$$\frac{B_t}{R_t} = h_t B_{t-1} + (\gamma - \tau_t) P_t y_t \tag{8}$$

with $h_t \in (0,1)$.

Fiscal policy: fiscal limit and rule of default.

We denote by $b_t = B_{t-1}/P_t y_t$, the real burden of *contractual debt* (or debt due) - what is owed by government, relative to nominal GDP at t - and by $\omega_t = h_t B_{t-1}/P_t y_t$, the real burden of *debt actually redeemed*, relative to nominal GDP, when the possibility of default is taken into account. We refer to ω_t as the effective debt ratio. Following Bi (2010), Daniel and Shiamptanis (2010) and Davig, Leeper and Walker (2011), we assume that the tax rate increases with the fraction of debt to GDP, up to an upper limit[2], denoted by $\hat{\tau}$. More precisely, we assume that the tax rate depends on the actual debt burden, ω_t, and thus:

$$\tau_t = \Upsilon(\omega_t) = \min\left(\bar{\tau} + \theta \cdot (\omega_t - \bar{\omega}); \hat{\tau}\right) \tag{9}$$

where θ and $\bar{\tau}$ satisfy:

$$\begin{aligned} \bar{\tau} &= \gamma + (1-\beta)\bar{\omega} < \hat{\tau}, \\ \theta &> 1-\beta. \end{aligned}$$

[2]The value $\hat{\tau} = (1+\sigma)/(1+2\sigma)$ is a natural candidate for this upper limit as it represents the highest point of the Laffer curve in this economy (se below).

The term $\bar{\omega}$ can be interpreted as a target value for the effective debt to GDP ratio. From (9), we define a level $\hat{\omega}$ at which the tax rate reaches its maximum:

$$\tau_t = \hat{\tau} \Longleftrightarrow \omega_t \geq \bar{\omega} + \frac{\hat{\tau} - \bar{\tau}}{\theta} \equiv \hat{\omega}. \tag{10}$$

Concerning default, we assume that government does not adopt a strategic behavior but applies a simple rule, contingent on the level of contractual debt b_t:

$$h_t = \mathcal{H}(b_t) \leq 1.$$

The specification of this rule will be discussed below.

Monetary policy.

The central bank follows a "conventional" monetary policy, which corresponds to the case where the central bank controls the riskless interest rate, and not the interest rate on public debt. Precisely, it follows a Taylor rule taking into account the positivity of the nominal interest rate:

$$R_t^f = \Phi\left(\frac{\pi_t}{\bar{\pi}}\right) = \max\left(\beta^{-1}\bar{\pi}\left(\frac{\pi_t}{\bar{\pi}}\right)^{\phi}; 1\right)$$

with $\phi \geq 1$.

2.3 Equilibrium conditions.

We assume all markets clear perfectly. Given the optimizing first-order conditions, the equilibrium is defined by the following set of equations:

$$\begin{aligned}
y_t &= \left(\frac{\delta}{1-\gamma}\right)^{\frac{\sigma}{1+\sigma}} (1-\tau_t)^{\frac{\sigma}{1+\sigma}} A_t && (11)\\
\frac{1}{R_t} &= \beta E_t \frac{h_{t+1} y_t}{\pi_{t+1} y_{t+1}} && (12)\\
\frac{1}{R_t^f} &= \beta E_t \frac{y_t}{\pi_{t+1} y_{t+1}} && (13)\\
\frac{\pi_{t+1} y_{t+1}}{y_t} \frac{b_{t+1}}{R_t} &= h_t b_t + \gamma - \tau_t && (14)\\
R_t^f &= \Phi\left(\frac{\pi_t}{\bar{\pi}}\right) && (15)\\
h_t &= \mathcal{H}(b_t) && (16)\\
\tau_t &= \Upsilon(h_t b_t) && (17)
\end{aligned}$$

in which variables y_t, R_t, R_t^f, π_t, τ_t, h_t, and b_t are unknown.

3 Public debt and inflation

Actually, by combining on the one hand the government budget constraint (14) with (12) and on the other hand the policy rule (15) with (13), and using $\omega_t = h_t b_t$, a rational expectations equilibrium satisfies the following set of equalities:

i) two dynamic conditions:

$$\begin{aligned} E_t\omega_{t+1} &= \beta^{-1}\omega_t + \beta^{-1}\left(\gamma - \Upsilon\left(\omega_t\right)\right) && (18) \\ 1 &= \frac{\beta}{\bar{\pi}} E_t \frac{\mathcal{Y}\left(\omega_t, A_t\right)}{\mathcal{Y}\left(\omega_{t+1}, A_{t+1}\right)} \frac{\Phi\left(\pi_t/\bar{\pi}\right)}{\pi_{t+1}/\bar{\pi}} && (19) \end{aligned}$$

with:

$$\begin{aligned} \mathcal{Y}\left(\omega_t, A_t\right) &\equiv \left(\delta \frac{1-\Upsilon\left(\omega_t\right)}{1-\gamma}\right)^{\chi} A_t && (20) \\ \Phi\left(\pi_t/\bar{\pi}\right) &\equiv \max\left(\beta^{-1}\bar{\pi}\left(\pi_t/\bar{\pi}\right)^{\phi}, 1\right) && (21) \\ \Upsilon\left(\omega_t\right) &\equiv \min\left(\bar{\tau} + \theta\cdot\left(\omega_t - \bar{\omega}\right); \hat{\tau}\right) && (22) \end{aligned}$$

and $\chi = \sigma/\left(1+\sigma\right)$.

ii) the rule of default, linking ω_t and b_t :

$$\omega_t = \mathcal{H}\left(b_t\right) b_t \tag{23}$$

iii) a condition linking b_t et π_t :

$$\mathcal{Y}\left(\mathcal{H}\left(b_t\right) b_t, A_t\right) \cdot b_t = \frac{1}{\pi_t} \frac{B_{t-1}}{P_{t-1}} \tag{24}$$

implying that these variables cannot be both predetermined,

iv) the transversality condition:

$$E_t \beta^{T-t} \omega_T = 0 \tag{25}$$

3.1 The public debt burden.

Using the fiscal rule (22) in equation (18), we get:

$$E_t\omega_{t+1} = \begin{cases} \left(1-\theta\right)\beta^{-1}\omega_t + \left(1-\left(1-\theta\right)\beta^{-1}\right)\bar{\omega} & \text{pour } \omega_t \leq \hat{\omega} \\ \beta^{-1}\omega_t - \beta^{-1}\left(\hat{\tau}-\gamma\right) & \text{pour } \omega_t > \hat{\omega} \end{cases} \tag{26}$$

where $\hat{\omega}$ is given by (10). From the first equation, ω_t converges toward a steady state $\omega = \bar{\omega}$, when $\left(1-\theta\right)\beta < 1$ (which we assume). The second equation is divergent, and admits a steady state defined by:

$$\omega = \frac{\hat{\tau}-\gamma}{1-\beta} \equiv \omega^{\sup} \tag{27}$$

ω^{sup} can be interpreted as a critical debt level beyond which the trajectory of expected *effective* debt - that is integrating the possibility of future default - is explosive. As $\bar{\omega} < \hat{\omega} < \omega^{\text{sup}}$, the debt dynamics may be represented by the following figure:

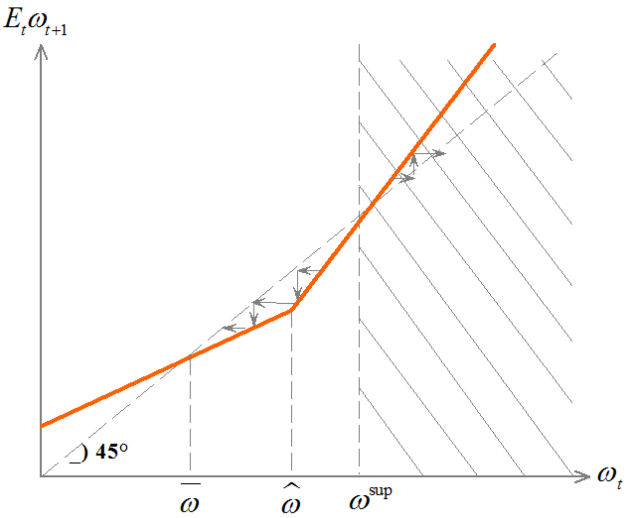

Figure1

For a given value of ω_t satisfying $\omega_t < \omega^{\text{sup}}$, the expected burden of effective debt converges toward $\bar{\omega}$. However remark that this dynamics includes the possibility of a future default and does not necessarily captures the actual dynamics of effective debt which depends on realisations of the productivity shock (see *infra*).

The case $\omega_t > \omega^{\text{sup}}$ is impossible. In order to understand this crucial result, it suffices to remark that equation (18) indeed summarizes the government budget constraint (14), but also the arbitrage condition (12) taking into account the possibility of default. As effective debt $\omega_t = h_t b_t$ includes a possible default on public debt, the condition $E_t\omega_{t+1} > \omega_t > \omega^{\text{sup}}$ is necessarily associated with an explosive dynamics which violates the transversality condition (25), or the rational expectations hypothesis, if it is stopped at some finite level. For this reason, ω^{sup} represents the maximum debt level, but also the *default threshold.*

3.2 Stationnary inflation.

Based on (19), we find two possible solutions for steady-state inflation. The first one, $\pi^* = \bar{\pi}$, corresponds to the case when the central bank does not meet the positivity constraint for the riskless interest rate and achieves the inflation target. The second one corresponds to a liquidity trap equilibrium, characterized by a zero riskless interest rate, which implies: $\pi^{**} = \beta$. Remark however that this second solution requires $\phi > 1$.

3.3 The joint dynamics of public debt and inflation in the absence of default.

By joining the results obtained in the two previous subsections, there exist four steady state equilibria (ω, π), that is: $(\bar{\omega}, \bar{\pi})$, $(\bar{\omega}, \beta)$, $(\omega^{\text{sup}}, \bar{\pi})$ and $(\omega^{\text{sup}}, \beta)$.[3] In appendix A, it is shown that the dynamics of this economy can be studied when linearizing the system (19) and (18) in the neighborhood of each steady state. The following representation obtains:

$$\begin{pmatrix} E_t\hat{\pi}_{t+1} \\ E_t\hat{\omega}_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{\pi}_t \\ \hat{\omega}_t \end{pmatrix} + \mathbf{B} \left(E_t \frac{A_{t+1}}{A_t} \right)$$

where $\hat{\omega}_t = \omega_t - \omega$, $\hat{\pi}_t = (\pi_t - \pi)/\bar{\pi}$, $\mathbf{A}$ is a (2×2) matrix and $\mathbf{B}$, a (2×1) column vector. We denote by λ_π and λ_ω the two eigenvalues of $\mathbf{A}$. We get $\lambda_\pi = \phi$ and $\lambda_\omega = (1-\theta)\beta^{-1}$ when the inflation target is hit, *i.e.* at $(\bar{\omega}, \bar{\pi})$ and $(\omega^{\text{sup}}, \bar{\pi})$, and $\lambda_\pi = 0$ and $\lambda_\omega = \beta^{-1}$ in the liquidity trap equilibria, *i.e.* at $(\bar{\omega}, \beta)$ and $(\omega^{\text{sup}}, \beta)$.

Given equation (24), the price level P_t is the unique non-predetermined variable when income is exogenous.[4] Assuming $h_t = 1\ \forall t$, one can apply the standard Blanchard et Kahn (1980) criteria in order to study the conditions for local uniqueness, i.e. the equilibrium determinacy. We then need to consider either ω_t or π_t comme non-predetermined and the other one as predetermined. A steady state equilibrium is locally uniquely stable if and only if one eigenvalue is, in absolute value, bigger than 1, and the other one, lower.[5]

Before pursuing the local analysis of steady states, it may be useful to offer a heuristic representation of inflation dynamics. Let us focus on the first line of the matrix system offered above, neglecting the productivity shock and the public debt effect. We get: $E_t\hat{\pi}_{t+1} = \mathbf{A}_{11}\hat{\pi}_t$. Expressing this equation in levels, and fulfilling the positivity constraint

[3] When $\phi < 1$, there exist only two such equilibria.

[4] This is the case when $\Upsilon(\omega_t) = \hat{\tau}$.

[5] Eigenvalues λ_π and λ_ω are always real.

on the riskless nominal interest rate, the following figure obtains when $\phi > 1$:

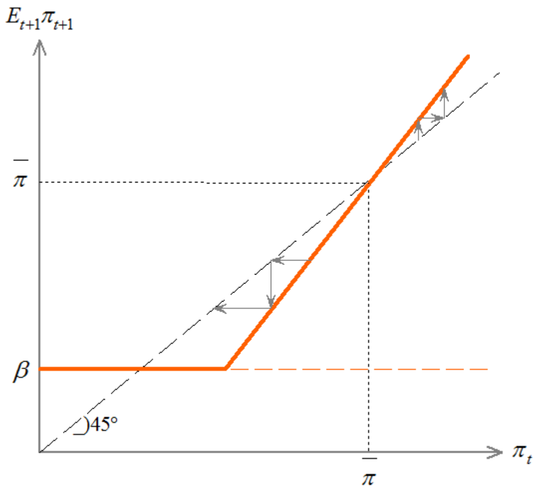

Figure2

Looking at figures 1 and 2 allows us intuitively check the effect of Blanchard and Kahn conditions on the more global dynamics of the economy.

At $(\bar{\omega}, \beta)$, both eigenvalues (0 and $(1-\theta)\,\beta^{-1}$) are inferior to unity. The steady state is locally non-determined. In Leeper's terms, fiscal and monetary policies are both locally passive. When $(1-\theta)\,\beta^{-1} < 1$, fiscal policy guarantees by a strong enough taxing rule, that public debt converges toward its target value, and is said locally passive. Monetary policy is passive because the riskless interest rate is equal to zero, the dangers of which have been recalled by Benhabib, Schmitt-Grohé et Uribe (2001).

At $(\bar{\omega}, \bar{\pi})$ and $(\omega^{\sup}, \beta)$, one eigenvalue is inferior to 1. The first equilibrium corresponds to the case commonly envisaged by economists. Fiscal policy is passive and monetary policy is active, ensuring a stability in the inflation rate.[6] The second equilibrium is more "exotic". It corresponds to a debt level equal to its maximum $\omega^{\sup}$ and a liquidity trap. It is an interesting application of the fiscal theory of the price level developped by Leeper (1991), Sims (1994), Woodford (1994, 1995) and Cochrane (2001). Interestingly, these two equilibria, locally determined, coexist in our economy. Let us assume that the economy is initially posited in on a trajectory converging toward the targeted equilibrium $(\bar{\omega}, \bar{\pi})$.[7] An expectation shock, possibly due to a negative productivity shock, has the capacity to make the economy jump in the neighborhood of $(\omega^{\sup}, \beta)$. An

[6] A full stabilization requires a more effective rule than a Taylor rule when the natural rate of interest fluctuates.

[7] In the absence of shocks, inflation would always be equal to $\bar{\pi}$ if the tax rate does not evolve with debt, or if non distorsive taxes did not affect natural income and interest rate.

important deflation will indeed imply a sudden decrease in the nominal interest rate and an increase in the real debt ratio.[8]

At $(\omega^{\sup}, \bar{\pi})$, both eigenvalues are higher than 1. The equilibrium is overdetermined, unstable. Woodford (1995) considers a similar situation in the case when the Central Bank controls money supply. The fiscal theory of the price level is still valid, if inflation follows a diverging trajectory, leading either to deflation or hyperinflation. Loyo (1999) and Blanchard (2005) use this type of argument to explain Brazilian inflation in the 1990's.

4 Default, risk of default and the dynamics of public debt.

In this section we introduce the possibility of default. The neighborhood of steady state $(\omega^{\sup}, \bar{\pi})$ offers a convenient analytical framework to understand the risk of default. Fiscal policy is passive, constrained by the impossibility to reduce expenditures (assumed to be proportional to income) or to increase fiscal receipts, as the tax rate has reached its upper limit $\hat{\tau}$. Monetary policy is active, aiming at limiting the variation of inflation around its target, thus forbidding prices to adjust and balance real debt with the present discounted value of future primary surpluses, as in the fiscal theory of the price level. Lastly, the initial level of debt is high, close enough to the default threshold $\omega^{\sup}$.

4.1 Default as a source of macroeconomic stability.

Uribe (2006) is the first to offer a (fiscale) theory of sovereing default in a monetary model. The initial situation is identical to the one analyzed by Woodford (1995), Loyo (1999) and Blanchard (2004). He shows that, by explicitly introducing the possibility of actual default, default occurs as an adjustment variable. In the presence of shocks, default is observed at each period, allowing inflation to remain at its target value and public debt not to overrun its maximum sustainable value (in our notation $\omega^{\sup}$). However in the case of positive shocks, a "negative" default occurs in this model. Lenders receive more than the contractual debt to be reimbursed... This feature is irrealistic and in addition, it nullifies any risk premium since positive default are as probable as negative default (assuming normal shocks). Interestingly, Uribe shows that if default is delayed (in the case of negative shocks), inflation may temporarily increase until default, the magnitude

[8]This "*Public Debt-Deflation*" mechanism is been studied by Aloui and Guillard(2011) in a more complex model with capital and wealth effects.

of which is then much more important than in the case of an immediate default.

Uribe's analysis relies on the importance of the role of the transversality condition in the economy he considers. In the absence of another steady state equilibrium (as $\bar{\omega}$ in our model), a permanently decreasing debt converging to a "low" steady state level is not more possible than a permanently increasing debt. Our model does not suffer from this shortcoming. Fiscal policy is obliged to be passive only locally, because of the existence of a high level of debt and a fiscal limit.

4.2 The unsustainable dynamics of public debt.

Let us now study the expected dynamics of contractual debt b_t (to be redeemed). It may differ from the expected dynamics of effective debt (actually redeemed) ω_t, because of the risk of default and its associated risk premium. We consider the case $\hat{\omega} < \omega_t < \omega^{\text{sup}}$, when the fiscal limit has been reached and the tax rate does not respond anymore to the debt ratio. We also assume that the Central Bank follows an active monetary policy and is able to control inflation around its targer, avoiding hyperinflationist and deflationist trajectories. Inflation is the non-explosive solution to equation (19) which rewrites in this case:

$$1 = E_t \frac{A_t}{A_{t+1}} \frac{(\pi_t/\bar{\pi})^{\phi}}{\pi_{t+1}/\bar{\pi}}$$

Under the assumption $\phi > 1$, and if the productivity growth factor[9] is autoregressive, then inflation is positively correlated with output growth.

Let us detail the dynamics of ω_t and b_t. From (26) and (27), we get for $\omega_t = h_t b_t$:

$$E_t \omega_{t+1} = (1+r)\,\omega_t - r\omega^{\text{sup}} \tag{28}$$

where $r = \beta^{-1} - 1$ represents the rate of preference for the present and ω^{sup} the steady state. Form the initial government budget constraint (14) and the definition of R_t^f given by (13), we can show (see B) that the dynamics of b_t can be written as:

$$\frac{E_t b_{t+1}}{1+\mathbf{p}_t} = (1+r)\,h_t b_t - r\omega^{\text{sup}} \tag{29}$$

where $\mathbf{p}_t = R_t/R_t^f$ is the risk premium associated with the holding of public debt.

When the risk premium is positive, *i.e.* $R_t > R_t^f$, the expected relative contractual debt $E_t b_{t+1}$ depends on its current value with a weight $(1+\mathbf{p}_t)(1+r)$ superior to $1+r$, which plays an equivalent role in equation (28).

[9] and income: $y_{t+1}/y_t = A_{t+1}/A_t$. See equation (20) when $\Upsilon(\omega_t) = \hat{\tau}$.

We define a "risky steady state" (RSS) without default[10] as satisfying $E_t b_{t+1} = b_t$, $h_t = 1$ and we assume that it is consistent with $\mathbf{p} > 0$. We find that, corresponding at this steady state:

$$b^{RSS} = \frac{(1+\mathbf{p})\,r}{(1+\mathbf{p})\,r + \mathbf{p}}\omega^{\sup}$$

That is the contractual debt ratio associated with the RSS is less than the default threshold $\omega^{\sup}$. It is easy to check that b^{RSS} is a decreasing function of the steady state risk premium $\mathbf{p}$. Noticing that $(1+\mathbf{p})(1+r) > 1$ corresponds to the slope of the dynamical equation which links $E_t b_{t+1}$ to b_t in the neighborhood of steady state b^{RSS}, this implies that this equilibrium is unstable. If, at t, b_t satisfies $b^{RSS} < b_t < \omega^{\sup}$, then in the absence of future shocks, the debt ratio will increase and reach the default threshold $\omega^{\sup}$, which will trigger default. The set $\left[b^{RSS}, \omega^{\sup}\right]$ represents a potential instability zone for public debt, i.e. a zone of unsustainability.

Notice that the definition of b^{RSS} In order to confirm this result, it is necessary to specify a default rule, which formalizes the rescheduling of publci debt. Let us assume the following rule:

Rule of default

$$\mathcal{H}(b_t) = \begin{cases} \mathbf{h} \cdot \omega^{\sup}/b_t < 1 & \text{if } b_t > \omega^{\sup} \\ 1 & \text{if not.} \end{cases} \tag{30}$$

with $0 \leq \mathbf{h} \leq 1$.

$\mathbf{h}$ is a control variable of this rule. According to this rule, any contractual debt level beyond the threshold $\omega^{\sup}$ triggers default and rescheduling. This rescheduling is such that the after-default effective debt ratio is a fraction of $\omega^{\sup}$, i.e.: $\omega_t = \mathcal{H}(b_t)\, b_t = \mathbf{h} \cdot \omega^{\sup}$. If we consider the limit case where the overrunis negligible ($b_t \rightarrow \omega^{\sup +}$), $\mathbf{h}$ can be interpreted as the maximum redemption rate. By extension, $1 - \mathbf{h}$ is the minimal rate of default - or adjustment rate.

The dynamics of contractual debt itself depends on the default rule, since this rule affects the risk premium and thus the price and quantity of the issued public bonds. The following proposition characterizes this link and summarizes the results sketched above.

Proposition 1 *Under rule $\mathcal{H}(\cdot)$, the risk premium $\mathbf{p}_t$ is an increasing function of the expected relative contractual debt $E_t b_{t+1}$, and a decreasing function of $\boldsymbol{h}$: $\mathbf{p}_t = p_t\left(\underset{+}{E_t b_{t+1}/\omega^{\sup}}; \underset{-}{\boldsymbol{h}}\right)$. The risky steady state (RSS) is such that: $b^{SS} < \omega^{\sup}$. The dynamics of contractual debt, b_t, is locally unstable in the neighborhood of b^{SS}.*

[10] See Appendix C.2.

Proof. *See Appendix C.* ■

The incidence of the adjustment rate $1 - \mathbf{h}$ in the debt dynamics is ambiguous. The higher this rate, the higher the correction, which moves away public debt from the default threshold and therefore decreases the probability of a future default. On the other hand, it increases the risk premium and hence the debt burden. It thus accelerates the growth of debt and makes it return toward the default threshold. This leads to a dilemma with respect to the adjustement rate. Actually this reflects the discussion on the quasi-default of Greece at the end of 2011, beginning of 2012: what is the sustainable debt level in Greece, which should be the target of the rescheduling? Defaulting does not suffice, it may be that the after-default situation is suche that the debt dynamics remains unsustainable and the economy heading toward a new default. The paradox is that there may be a self-reinforcing mechanism when the parameter $\mathbf{h}$ is known (or expected) by lenders. The expectation of a future default lead lenders to ask for high risk premias which increase the debt burden and thus set public debt on an unsustainable trajectory leading to renewed default.

This leads us to think that a "successful default" is such that debt dynamics is corrected "for good" and becomes sustainable. For the default rule assumed above, can we get a "successful default"? The following proposition answers this question.

Proposition 2 *There exists critical value* $\boldsymbol{\delta}$**,** *such that: if* $\boldsymbol{h} < \boldsymbol{\delta}$ *, in the case of default in* t, $h_t < 1$, *and in the absence of future shocks, we get:* $h_{t+s} = 1, \forall s > 0$, *and public debt converges toward* $\overline{\omega}$.

Proof. *See Appendix D.* ■

Proposition 2 gives us a condition on the default rule such that a default at t, $h_t < 1$, generates a sufficient decrease in the effective debt (a sufficient rescheduling) so that its future dynamics in the absence of new shocks, gradually moves it away from the risky zone. This condition is that the adjustment (rescheduling) coefficient of public debt be sufficiently high. When this is the case, the real debt burden becomes sustainable again, in particular because the risk premium decreases, which avoids the self-reinforcing effect due to the expectation of future default. This proposition meets intuition: when a state defaults on its sovereign debt, the rescheduling (i.e. reduction) of debt must sufficiently large to avoid a repetition of the default cycle.

5 Conclusion

The issue of default is complex when the dynamics of public debt depends on current and future macroeconomic shocks and also on the interaction between fiscal and monetary policies which may be incompatible. Moreover it is crucial to bear in mind the role of expected default and hence of risk premiums in the (un)sustainability of public debt. In this paper, using a dynamic macromodel, we tackle this issue and explicitly take into account all these effects.

In this paper we show that it is useful to distinguish between the “default” and the “no-default” thresholds. The default threshold corresponds to the notion of an upper debt limit or ceiling. Default occurs when lenders do not expect that government will be able to honor its debt. The no-default threshold is defined as a critical value of the debt ratio below which, in the absence of future shocks public debt will converge to its steady state level. When the debt ratio is between these two values, lenders still affect a positive probability to a full redemption of debt, but request a risk premium so high that the debt burden is too heavy and increasing, leading to default, in the absence of future favorable shocks.

We prove that the rule of default, that is the mode of rescheduling debt after default plays a crucial role in the characterization of both thresholds since it conditions the risk premiums to be applied to public debt. In this perspective a rule of default plays an ambiguous role in the default process. We then show that, in order to obtain a “successful default”, that is a default leading after rescheduling to the steady state debt level, the rule of default must be sufficiently aggressive: the rescheduled debt must be set below the no-default threshold.

These results lead to interesting issues, such at the time-consistency of rules of default and on the optimal timing of default procedures. The role of macroeconomic policies, including “unconventional” monetary policies, targeting the refinaning cost of government deficits, is also to be addressed. We leave these issues to future research.

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Appendix

A Linearizing the model

Here we linearize the system formed of equations (18) and (19), using (21) and (22) around a given steady state (ω, π). We define $\hat{\omega}_t = \omega_t - \omega$, and $\hat{\pi}_t = (\pi_t - \pi)/\bar{\pi}$. We find:

$$\begin{aligned}
E_t\hat{\pi}_{t+1} &= \varepsilon(\pi/\bar{\pi}) \cdot \hat{\pi}_t - \frac{\pi}{\bar{\pi}}\left(E_t\frac{A_{t+1}}{A_t} - 1\right) \\
&\quad + \chi\frac{\Upsilon'(b)}{1-\Upsilon(b)}\left([1-\Upsilon'(b)]\beta^{-1} - 1\right)(\pi/\bar{\pi})\hat{\omega}_t \\
E_t\hat{\omega}_{t+1} &= (1-\Upsilon'(b))\beta^{-1}\hat{\omega}_t
\end{aligned}$$

where $\varepsilon(\pi/\bar{\pi}) = (\pi/\bar{\pi})\Phi'(\pi/\bar{\pi})/\Phi(\pi/\bar{\pi})$ represents the elasticity of $\Phi(\cdot)$ at the steady state. Using a matrix notation, we get:

$$\begin{pmatrix} E_t\hat{\pi}_{t+1} \\ E_t\hat{\omega}_{t+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} \hat{\pi}_t \\ \hat{\omega}_t \end{pmatrix} + \mathbf{B}\left(E_t\frac{A_{t+1}}{A_t} - 1\right)$$

with:

$$\mathbf{A} = \begin{pmatrix} \varepsilon(\pi/\bar{\pi}) & \chi\frac{\Upsilon'(b)}{1-\Upsilon(b)}\left([1-\Upsilon'(b)]\beta^{-1} - 1\right)\pi/\bar{\pi} \\ 0 & (1-\Upsilon'(b))\beta^{-1} \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} \pi/\bar{\pi} \\ 0 \end{pmatrix}$$

The two eigenvalues of $\mathbf{A}$ are:

$$\begin{aligned}
\lambda_\pi &= \varepsilon(\pi/\bar{\pi}), \\
\lambda_\omega &= (1-\Upsilon'(b))\beta^{-1}
\end{aligned}$$

Remark that $\varepsilon(\pi/\bar{\pi}) = \phi$ at $\pi = \bar{\pi}$ and $\varepsilon = 0$ at $\pi = \beta$, $\Upsilon'(b) = \theta$ at $b = \bar{\omega}$ and $\Upsilon'(b) = 0$ at $b = \omega^{\sup}$. Using these equalities, for the valuation of eigenvalues λ_π and λ_ω for steady states: $(\bar{\omega}, \bar{\pi})$, $(\bar{\omega}, \beta)$, $(\omega^{\sup}, \bar{\pi})$ and $(\omega^{\sup}, \beta)$, we find:

$$\begin{pmatrix} \lambda_\pi \\ \lambda_\omega \end{pmatrix} \in \left\{ \begin{pmatrix} \phi \\ (1-\theta)\beta^{-1} \end{pmatrix}_{(\bar{\omega},\bar{\pi})}, \begin{pmatrix} 0 \\ (1-\theta)\beta^{-1} \end{pmatrix}_{(\bar{\omega},\beta)}, \begin{pmatrix} \phi \\ \beta^{-1} \end{pmatrix}_{(\omega^{\sup},\bar{\pi})}, \begin{pmatrix} 0 \\ \beta^{-1} \end{pmatrix}_{(\omega^{\sup},\beta)} \right\}$$

B Debt dynamics and risk premium

From the government budget constraint (14), we get when $\tau_t = \hat{\tau}$:

$$\frac{\pi_{t+1}y_{t+1}}{y_t}\frac{b_{t+1}}{R_t} = h_t b_t - (\hat{\tau} - \gamma) \tag{B.1}$$

Using the definition of the riskless interest rate given by (13):

$$1/R_t^f = \beta E_t \frac{y_t}{\pi_{t+1} y_{t+1}},$$

the left-handside of equation (B.1) can be rewritten as:

$$\begin{aligned} \frac{\pi_{t+1} y_{t+1}}{y_t} \frac{b_{t+1}}{R_t} &= R_t^f \left(\beta E_t \frac{y_t}{\pi_{t+1} y_{t+1}} \right) \frac{\pi_{t+1} y_{t+1}}{y_t} \frac{b_{t+1}}{R_t} \\ &= \frac{R_t^f}{R_t} \beta E_t \left(\frac{y_t}{\pi_{t+1} y_{t+1}} \frac{\pi_{t+1} y_{t+1}}{y_t} b_{t+1} \right) = \beta \frac{R_t^f}{R_t} E_t b_{t+1} \end{aligned}$$

Introducing this expression in (B.1), multiplying each side with β^{-1} and denoting $\mathbf{p}_t = R_t / R_t^f - 1$, we eventually get:

$$\frac{E_t b_{t+1}}{1 + \mathbf{p}_t} = (1 + r) h_t b_t - r \omega^{\text{sup}} \tag{B.2}$$

also using (27) and $r = \beta^{-1} - 1$.

C Proof of Proposition 1

We define δ_{t-1} as:

$$\delta_{t-1} = \frac{B_{t-1}}{\bar{\pi} P_{t-1} y_{t-1}} / \omega^{\text{sup}}$$

δ_{t-1} is predetermined. It corresponds to the ratio of public debt to the GDP, contracted in $t-1$, after correction with steady state inflation, and divided by the upper debt limit ω^{sup}. Correcting by $\bar{\pi}$ allows us to compare the numerator with $b_t = B_{t-1}/P_t Y_t$: at steady state, we get: $\delta = b/\omega^{\text{sup}}$. Using the notation:

$$\alpha_t = \bar{\pi} \frac{P_{t-1} y_{t-1}}{P_t y_t} = \frac{\bar{\pi}}{\pi_t} \frac{y_{t-1}}{y_t},$$

we get:

$$b_t = \alpha_t \delta_{t-1} \omega^{\text{sup}} \tag{31}$$

Hence the ratio of contractual debt, to be reimbursed in t, to GDP is equal to the product of the ratio of this debt to the previous period GDP, multiplied by the inverse of the growth factor of nominal GDP at t, evaluated at the steady state growth rate. Under the assumption made about inflation in section (4.2), α_t may be considered as exogenous and inversely correlatd with the growth factor of productivity. Default happens when $b_t > \omega^{\text{sup}}$, that is as soon as: $\alpha_t > 1/\delta_{t-1}$.

C.1 Computing risk premium $\mathbf{p}_t$

From (12) and (13), one gets the following expression for the risk premium $\mathbf{p}_t = R_t/R_t^f - 1$, depending on the expected future values of α_{t+1} :

$$\mathbf{p}_t = \frac{E_t\alpha_{t+1}}{E_t h_{t+1}\alpha_{t+1}} - 1$$

Denoting by $G_t(\cdot)$ the distribution function in period t of α_{t+1} and using the default rule (30), we get:

$$\mathbf{p}_t = \frac{\Delta_t(\delta_t; \mathbf{h})}{E_t\alpha_{t+1} - \Delta_t(\delta_t; \mathbf{h})} \tag{C.2}$$

with:

$$\Delta_t\left(\underset{+}{\delta_t}; \underset{-}{\mathbf{h}}\right) \equiv \int_{1/\delta_t} [\alpha - \mathbf{h}/\delta_t] \cdot dG_t(\alpha) \tag{C.3}$$

where signs under the function arguments $\Delta_t(\cdot)$ correspond to the sign of partial derivatives.

Remark that if there exists a value $\alpha_{t+1}^{\sup}$ such that $\alpha_{t+1} \leq \alpha_{t+1}^{\sup}$ and $\delta_t < 1/\alpha_{t+1}^{\sup}$ (no possible default in $t+1$), then we get: $\mathbf{p}_t = \Delta_t(\delta_t; \mathbf{h}) = 0$.

From (31), we obtain:

$$\delta_t = \frac{E_t b_{t+1}}{E_t\alpha_{t+1}\omega^{\sup}}$$

Using this expression in the risk premium formula, we eventually get:

$$\mathbf{p}_t = \frac{\Delta_t(E_t b_{t+1}/E_t\alpha_{t+1}\omega^{\sup}; \mathbf{h})}{E_t\alpha_{t+1} - \Delta_t(E_t b_{t+1}/E_t\alpha_{t+1}\omega^{\sup}; \mathbf{h})} \equiv p_t(E_t b_{t+1}/\omega^{\sup}; \mathbf{h})$$

C.2 Dynamics and risky steady state (RSS)

Using (31), (B.1) may be rewritten as:

$$\frac{E_t\alpha_{t+1}\delta_t}{1+\mathbf{p}_t} = (1+r)\, h_t\alpha_t\delta_{t-1} - r$$

Replacing $\mathbf{p}_t$ by its expression given by (C.2), we get:

$$[E_t\alpha_{t+1} - \Delta_t(\delta_t; \mathbf{h})] \cdot \delta_t = (1+r)\, h_t\alpha_t\delta_{t-1} - r \tag{32}$$

whose implicit solution can be denoted by:

$$\delta_t = \Psi_t\left((1+r)\, h_t\alpha_t\underset{+}{\delta_{t-1}} - r; \underset{-}{\mathbf{h}}\right) \tag{C.5}$$

We define a risky steady state" (RSS) without default by using in equation (32): $\delta_t = \delta_{t-1} = \delta^{RSS}$, $h_t = 1$, as well as $\alpha_t = 1$, $E_t\alpha_{t+1} = 1$ and $\Delta_t(\cdot) = \Delta(\cdot)$. After rearranging terms, we get:

$$1 = \left[1 + \Delta\left(\underset{+}{\delta^{RSS}}; \underset{-}{\mathbf{h}}\right)/r\right] \cdot \delta^{RSS} \tag{33}$$

with $\Delta\left(\delta^{RSS};\mathbf{h}\right)>0$.

From the previous equation the steady state value δ^{RSS} obtains, which can be expressed as an implicit function of $\mathbf{h}$:

$$\delta^{RSS}=d\left(\mathbf{h}\right)$$

with:

$$\frac{\partial d\left(\mathbf{h}\right)}{\partial\mathbf{h}}=-\frac{\Delta'_{\mathbf{h}}\left(\delta^{RSS};\mathbf{h}\right)}{r/\left(\delta^{RSS}\right)^{2}+\Delta'_{\delta}\left(\delta^{RSS};\mathbf{h}\right)}>0$$

Since $\Delta\left(\delta^{RSS};0\right)=\int_{1/\delta^{SS}}\alpha\cdot dG\left(\alpha\right)>0$ and $\Delta\left(\delta^{RSS};1\right)=\int_{1/\delta^{SS}}\left[\alpha-1/\delta^{RSS}\right]\cdot dG\left(\alpha\right)>0$, we easily check the following results:

$$0<d\left(0\right)<d\left(1\right)<1$$

Let us reexamine dynamic equation (32) in the case $\alpha_t=1$ and $\delta_{t-1}<1$, which necessarily implies $h_t=1$. We get:

$$\left[1-\Delta\left(\delta_t;\mathbf{h}\right)\right]\cdot\delta_t=\left(1+r\right)\delta_{t-1}-r$$

Hence the steady state $\delta^{RSS}=d\left(\mathbf{h}\right)<1$. In its neighborhood, we get:

$$\left.\frac{\partial\delta_t}{\partial\delta_{t-1}}\right|_{\delta^{SS}}=\frac{1+r}{1-\Delta\left(\delta^{RSS};\mathbf{h}\right)-\Delta'_{\delta}\left(\delta^{RSS};\mathbf{h}\right)\cdot\delta^{RSS}}>1$$

The RSS equilibrium is unstable. Let us assume that there is no shock in t $\left(\alpha_t=1\right)$. We get: $b_t=\alpha_t\delta_{t-1}\omega^{\sup}=\delta_{t-1}\omega^{\sup}$. Let us assume that δ_{t-1} (resp. b_t) satisfies: $\delta^{RSS}<\delta_{t-1}<1$ (resp. $b^{RSS}<b_t<\omega^{\sup}$ with $b^{RSS}=\delta^{RSS}\omega^{\sup}$), then, in the absence of new shocks, the ratio of debt to GDP increases above $\omega^{\sup}$, which triggers default. On the other hand, if the debt ratio satisfies $b_t<b^{RSS}$, the ration of debt to GDP converges, in the absence of further shock, to its targeted steady state: $\bar{\omega}$.

D Proof of Proposition 2

Let us search for the condition implying $\mathbf{h}<\delta^{RSS}=d\left(\mathbf{h}\right)$. We look for the critical value$\boldsymbol{\delta}$, such that: $\boldsymbol{\delta}=d\left(\boldsymbol{\delta}\right)$. From equation (33), we get:

$$\boldsymbol{\delta}=\frac{r}{r+\Delta\left(\boldsymbol{\delta};\boldsymbol{\delta}\right)}\tag{D.1}$$

with: $\Delta\left(\boldsymbol{\delta};\boldsymbol{\delta}\right)=\int_{1/\boldsymbol{\delta}}\left[\alpha-1\right]\cdot dG\left(\alpha\right)$ and

$$\frac{\partial\Delta\left(\boldsymbol{\delta};\boldsymbol{\delta}\right)}{\partial\boldsymbol{\delta}}=\frac{1-\boldsymbol{\delta}}{\boldsymbol{\delta}}g\left(1/\boldsymbol{\delta}\right)>0.$$

The left-hand side of equation (D.1) is decreasing, and the right-hand side is increasing. Thus the solutions, if it exists is unique. As $0 < \Delta(1;1) = \int_1 [\alpha - 1] \cdot dG(\alpha) < E(\alpha) = 1$, the solution exists and is unique.

$d(\mathbf{h})$ has been proven to be an increasing function satisfying: $0 < d(0) < d(1) < 1$. It implies $\mathbf{h} < d(\mathbf{h})$ if $\mathbf{h} < \boldsymbol{\delta}$ and $\mathbf{h} > d(\mathbf{h})$ if $\mathbf{h} > \boldsymbol{\delta}$.